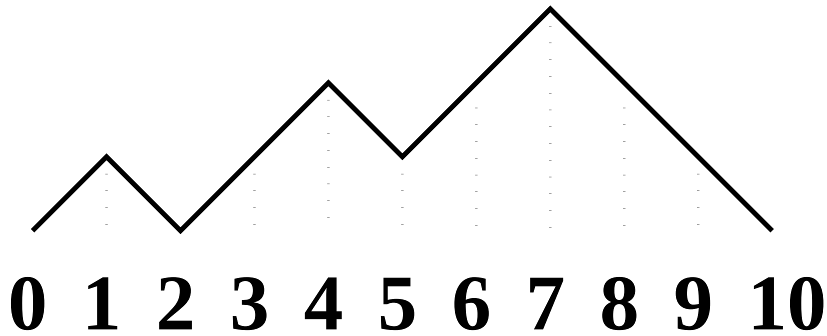

Fig. 1: A sample walk diagram of order 10.

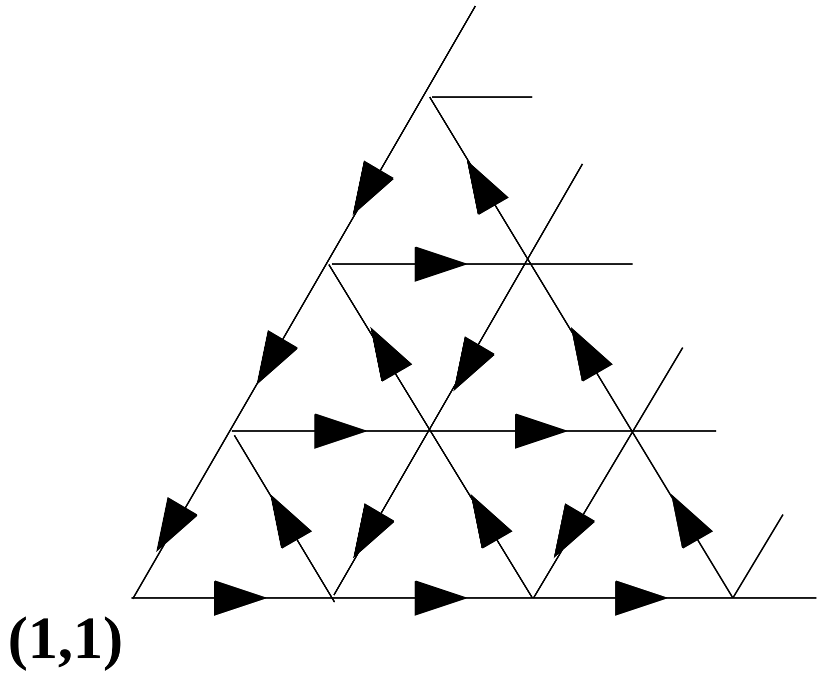

Fig. 2: The simplex Π_+. The three oriented links correspond respectively to ϵ_1 (right), ϵ_2 (up, left) and ϵ_3 (down, left). We have also indicated the origin $(1,1)$.

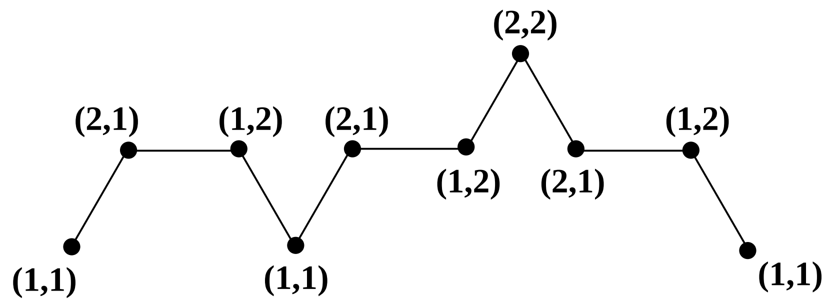

Fig. 3: A sample walk diagram of order 9. We have indicated by dots the successive weights visited by the path.

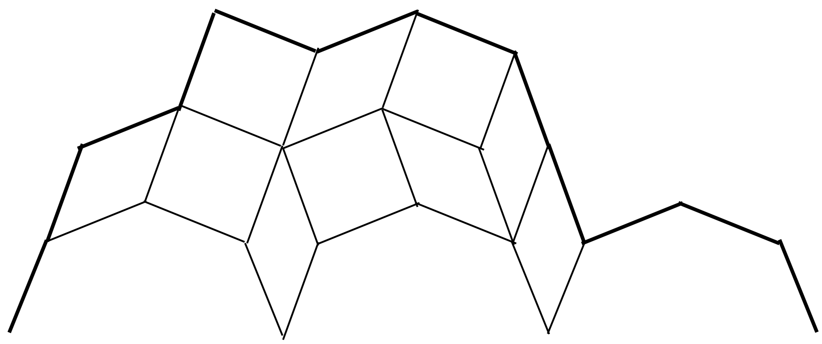

Fig. 4: A sample $SU(4)$ walk diagram a of order $4\times 3=12$ is represented in thick line. It is made of a succession of steps of the form (4.5). We have indicated its box decomposition in thin lines, leading from the fundamental diagram $a_0^{(4)}$ of order 12 to a, after a total of $|a|=10$ box additions.

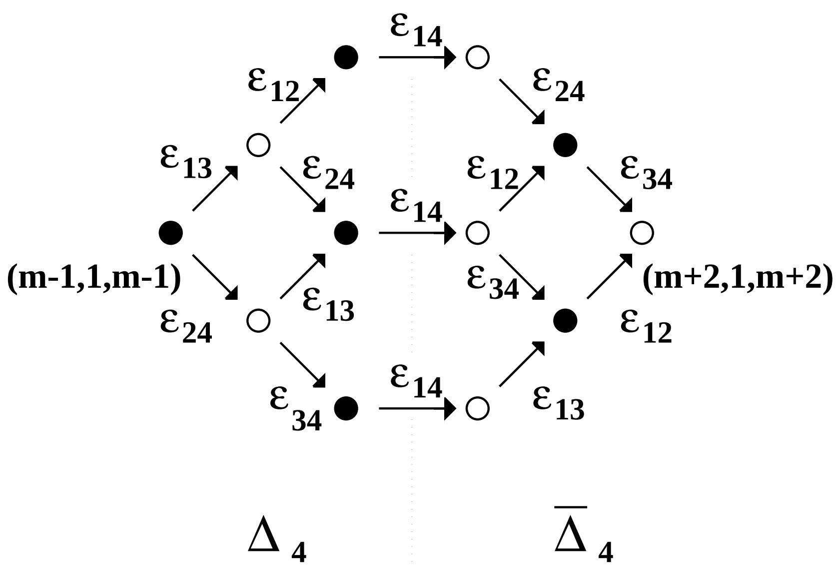

Fig. 5: The set of 12 vectors forming the difference operator defining $a_{m,n}^{(N)}$ in terms of $C_\Lambda^{(Nn)}$, for $N = 4$. We have indicated the vectors $\epsilon_{i,j} \equiv \epsilon_i - \epsilon_j$ linking the dots, representing $(m-1, 1, m-1) + u_k + v_l$ on the left half (Δ_4 operator), and $(m+2, 1, m+2) - \bar{u}_k - \bar{v}_l$ on the right half ($\bar{\Delta}_4$ operator). The terms with a filled black circle come with a $+$, those with an empty circle with a $-$ in the final difference operator.

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SU(N) Meander Determinants

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We propose a generalization of meanders, i.e., configurations of non-selfintersecting loops crossing a line through a given number of points, to $SU(N)$. This uses the reformulation of meanders as pairs of reduced elements of the Temperley-Lieb algebra, a $SU(2)$-related quotient of the Hecke algebra, with a natural generalization to $SU(N)$. We also derive explicit formulas for $SU(N)$ meander determinants, defined as the Gram determinants of the corresponding bases of the Hecke algebra.

02/97

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1. Introduction

In this paper we propose various generalizations of the concept of meander [1] [2] [3] [4] [5]. The original meander problem consists in counting the number M_n of meanders of order n, i.e. of topologically inequivalent configurations of a closed non-self-intersecting loop crossing an infinite line through $2n$ points. One can also define the corresponding multi-component meander problem, by demanding that the loop be replaced by a given number of non-intersecting loops (connected components). The meander problem probably first arose in the work of Poincaré about differential geometry, then reemerged in various contexts, such as the classification of 3-manifolds [6], or the physics of compact polymer folding [7].

In the present paper, we extend the purely algebraic approach advocated in [8], which relates multi-component meanders to *pairs* of reduced elements of the Temperley-Lieb algebra [9] (see also P. Martin's book [10] for an elementary introduction), or ideals thereof. The idea is to define generalized multi-component meanders as *pairs* of reduced elements of the $SU(N)$ quotients of the Hecke algebra [12] which generalize the Temperley-Lieb $SU(2)$ quotient, or of ideals thereof. The notion of "component" for generalized meanders still awaits a good combinatorial interpretation. We trade it in the present approach for a piece of information on any given generalized meander, provided by the Markov trace of the corresponding product of reduced elements. Given a reduced basis of the above Hecke algebra quotients or ideals, this information is summarized by the Gram matrix of the basis. The aim of this work is to compute explicitly the "meander determinants" namely the determinants of these Gram matrices.

The paper is organized as follows. In Sect.2, we recall basic definitions and summarize the results obtained in [8] and [11] for the $SU(2)$ meander determinant, in the form of an explicit determinantal formula. We also present the $SU(N)$ quotients of the Hecke algebra, generalizing the Temperley-Lieb algebra. In Sect.3, we focus our attention on the $SU(3)$ case. We are led to the natural definition of $SU(3)$ meanders, as pairs of elements of the reduced basis of a certain ideal $\mathcal{I}_{3n}^{(3)}(\beta)$ of the $SU(3)$ quotient $H_{3n}^{(3)}(\beta)$ of the Hecke algebra. This basis is labelled by closed paths of length $3n$ on the Weyl chamber of $sl(3)$, the $SU(3)$ walk diagrams. We then compute the corresponding Gram determinant, by direct orthogonalization of the basis. We obtain an explicit formula for the $SU(3)$ meander determinant. This result is generalized to $SU(N)$ in Sect.4, where we also establish a duality relation between the ideals $\mathcal{I}_{Nk}^{(N)}(\beta)$ and $\mathcal{I}_{Nk}^{(k)}(\beta)$, relating the

$SU(N)$ and $SU(k)$ meander determinants. In Sect.5, we derive a determinant formula for the Gram matrix of a reduced basis of the whole $SU(N)$ quotient $H_n^{(N)}(\beta)$ of the Hecke algebra. This coincides with the meander determinant only in the $SU(2)$ case, and suggests another possible generalization of meanders. We gather a few concluding remarks in Sect.6.

2. Meanders and SU(2)

2.1. Definitions

The meander problem of order $2n$ is that of enumerating the topologically inequivalent configurations of a planar non-intersecting closed road (loop) crossing a river (line) through $2n$ distinct bridges. A meander is therefore represented as a non-self-intersecting loop crossing a line through $2n$ distinct points. The line cuts the meander into an upper and a lower part, which are both made of n non-intersecting arches (pieces of the loop) connecting the $2n$ bridges by pairs. Such an upper (or lower) configuration of a meander is called an arch configuration of order $2n$. The set of arch configurations of order $2n$, A_{2n}, has cardinal equal to the Catalan number

$$c_n = \frac{(2n)!}{(n+1)!n!} \tag{2.1}$$

readily proved by induction.

n	1	2	3	4	5	6	7	8	9	10
c_n	1	2	5	14	42	132	429	1430	4862	16796

Table I: The Catalan numbers for $n = 1, 2, ..., 10$.

A multi-component meander of order $2n$ is the superposition of two arbitrary upper and lower arch configurations a, $b \in A_{2n}$. This results a priori in a configuration of k different non-intersecting roads crossing the river through a total of $2n$ bridges: k is called the number of connected components of the meander, also denoted by $k = \kappa(a, b)$.

We choose to adopt an alternative description of meanders in terms of $SU(2)$ walk diagrams as follows. A $SU(2)$ walk diagram of order $2n$ is a closed path of length $2n$ on the semi-infinite line $\{1, 2, 3, ...\}$ identified with the Weyl chamber of the $sl(2)$ Lie algebra.

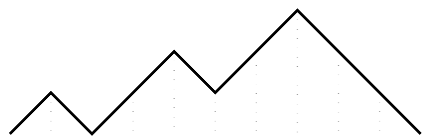


Fig. 1: A sample walk diagram of order 10.

More precisely, a walk diagram is a sequence $\{h(i),\ i = 0, 1, 2, ..., 2n\}$ of positive integer "heights", such that

$$h(i+1) - h(i) \in \{1, -1\} \qquad h(0) = h(2n) = 1 \tag{2.2}$$

A pictorial representation for a walk diagram is presented in Fig.1: it consists of the graph of the corresponding function $i \to h(i)$, whose points are joined by consecutive segments. We denote by $W_{2n}^{(2)}$ the set of walk diagrams[1] of order $2n$.

The walk diagrams of order $2n$ are in one-to-one correspondence with the arch configurations of order $2n$. Starting from an arch configuration of order $2n$ let us label by 0, 1, 2, ..., $2n$ respectively the portions of river to the left of the leftmost bridge, between the first and second, ..., to the right of the rightmost bridge along the river. We define the map $i \to h(i)$ by assigning to the portion of river labelled i the number $h(i)$ of arches passing above it, plus one[2].

The constraints (2.2) are satisfied by h hence we have constructed a walk diagram for each arch configuration; the process is clearly bijective, as an arch configuration is entirely determined by the numbers $h(i+1) - h(i)$, with the value $+1$ if an arch originates from the left bridge of the portion i of river and passes above it, and the value -1 if an arch terminates at the left bridge of the portion i of river (and therefore does not pass over it).

A multi-component meander of order $2n$ is therefore equivalently given by a couple (a, b) of walk diagrams of order $2n$, and we still denote by $\kappa(a, b)$ its number of connected components of road.

[1] Here and in the following, the superscript (2) stands for $SU(2)$.

[2] This is slightly different from the conventions of refs.[8] and [11], in which $h(i) \geq 0$ rather than 1. Our present choice is motivated by the form of the forthcoming $SU(N)$ and Hecke generalizations.

2.2. Temperley-Lieb algebra

The link between the above arch configuration and walk diagram pictures is provided by the Temperley-Lieb algebra, as well as a direct interpretation of the quantity $\kappa(a,b)$, for $a, b \in W_{2n}^{(2)}$.

The Temperley-Lieb algebra $TL_n(\beta)$ is defined by generators $1, e_1, e_2, ..., e_{n-1}$ and relations

$$\begin{aligned} e_i^2 &= \beta\, e_i \quad \text{for } i = 1, 2, ..., n-1 \\ e_i e_j &= e_j e_i \quad \text{for } |i-j| > 1 \\ e_i e_{i\pm 1} e_i &= e_i \quad \text{for } i = 1, 2, ..., n-1 \end{aligned} \tag{2.3}$$

An element of this algebra is said to be reduced if it is written as a product of generators, with a minimal number of them ("reduction" is achieved by repeated use of the relations (2.3)).

For reasons which will become clear later, we will work with a certain left ideal of the Temperley-Lieb algebra $TL_{2n}(\beta)$, which is however isomorphic to $TL_n(q)$. We denote by $\mathcal{I}_{2n}^{(2)}(\beta)$ the left ideal generated by the element $e_1 e_3 e_5 ... e_{2n-1}$ of $TL_{2n}(\beta)$.

There is a one-to-one correspondence between the reduced elements of the ideal $\mathcal{I}_{2n}^{(2)}(\beta)$ and the walk diagrams of order $2n$. To best see this, let us first reconsider the walk diagrams of order $2n$. We start from the "fundamental" walk diagram $a_0^{(2)} \in W_{2n}^{(2)}$, such that

$$h(1) = h(3) = ... = h(2n-1) = 2 \quad \text{and} \quad h(0) = h(2) = ... = h(2n) = 1 \tag{2.4}$$

This is the walk with the smallest values of the height $h(i)$. Now any other walk diagram of order $2n$ may be constructed by successive "box additions" on $a_0^{(2)}$. By box addition on a walk diagram a at position i, which we denote by $a + \diamond_i$, we mean the following transformation. For the box addition to be possible, a must have a minimum at the position i, namely $h(i+1) = h(i-1) = h(i) + 1$. The box addition then simply amounts to transform this minimum into a maximum, namely change $h(i) \to h(i) + 2$, without altering the other values of h. By successive box additions on $a_0^{(2)}$, it is easy to describe all the set of walk diagrams of order $2n$. Note that a given walk diagram may be obtained by distinct sequences of box additions on $a_0^{(2)}$, but all of them will consist of the same total number of box additions. We are now in position to construct a map φ from $W_{2n}^{(2)}$ to a basis of reduced elements of $\mathcal{I}_{2n}^{(2)}(\beta)$. We start with

$$\varphi(a_0^{(2)}) \;=\; e_1 e_3 ... e_{2n-1} \tag{2.5}$$

and proceed recursively, using box additions, by setting

$$\varphi(a+\diamond_i) \;=\; e_i\,\varphi(a) \tag{2.6}$$

The map is well-defined, as two distinct sequences of box additions leading to the same walk diagram correspond to different products of the same commuting e_i's (at each step, if two distinct box additions are possible, they take place at positions i and j with $|j-i|>1$, hence the corresponding e_i and e_j commute, due to (2.3)). It exhausts all the reduced elements of $\mathcal{I}_{2n}^{(2)}(\beta)$, which has the dimension c_n (2.1) as a vector space.

A meander is therefore equivalently given by a pair of reduced elements of $\mathcal{I}_{2n}^{(2)}(\beta)$. The Temperley-Lieb algebra $TL_n(\beta)$ is endowed with a natural scalar product attached to the Markov trace, denoted by Tr. The latter is defined by the normalization $\mathrm{Tr}(1)=\beta^n$, and the Markov property that for any element $E(e_1,e_2,...,e_{j-1})$ involving only e_i's with $i<j$, we have

$$\mathrm{Tr}\big(E(e_1,e_2,...,e_{j-1})e_j\big) \;=\; \eta\mathrm{Tr}\big(E(e_1,e_2,...,e_{j-1})\big) \tag{2.7}$$

The standard choice for $TL_n(\beta)$ for the constant η is

$$\eta \;=\; \frac{1}{\beta} \tag{2.8}$$

The trace extends linearly to any element of $TL_n(\beta)$. We also need to define the transposed e^t of an element $e\in TL_n(\beta)$, as $1^t=1$, $e_i^t=e_i$ for $i=1,2,...,n$, and $(ef)^t=f^te^t$ for any two elements $e,f\in TL_n(\beta)$; again, the definition extends to any element by linearity. This leads to the scalar product

$$(e,f) \;=\; \mathrm{Tr}(ef^t) \tag{2.9}$$

Remarkably, when restricted to the ideal $\mathcal{I}_{2n}^{(2)}(\beta)$, and when expressed between two reduced elements say $\varphi(a)$ and $\varphi(b)$, a, b two walk diagrams of order $2n$, this scalar product reads

$$(\varphi(a),\varphi(b)) \;=\; \beta^{\kappa(a,b)+n} \tag{2.10}$$

thus making the contact with our initial road/river picture of meanders. Defining the normalized reduced basis elements $(a)_1=\beta^{-n/2}\varphi(a)$ (this basis is referred to as basis 1 in the following), we have

$$\big((a)_1,(b)_1\big) \;=\; \beta^{\kappa(a,b)} \tag{2.11}$$

2.3. Meander determinant

The meander determinant $\Delta_{2n}^{(2)}(\beta)$ is defined as the determinant of the Gram matrix of the basis 1 above, namely the $c_n \times c_n$ matrix $\mathcal{G}_{2n}^{(2)}(\beta)$ with entries

$$\left[\mathcal{G}_{2n}^{(2)}(\beta)\right]_{a,b} = \beta^{\kappa(a,b)} \tag{2.12}$$

which therefore carries information about the multi-component meanders.

In [8] [11], we have derived an exact formula for $\Delta_{2n}^{(2)}(\beta)$ based on the explicit Gram-Schmidt orthogonalization of the matrix $\mathcal{G}_{2n}^{(2)}(\beta)$. The formula reads

$$\Delta_{2n}^{(2)}(\beta) = \prod_{m=1}^{n} \left[U_m(\beta)\right]^{a_{m,n}^{(2)}} \tag{2.13}$$

where $U_m(\beta)$ are the Chebishev polynomials of the first kind, with

$$U_m(2\cos\theta) = \frac{\sin(m+1)\theta}{\sin\theta} \tag{2.14}$$

and

$$a_{m,n}^{(2)} = C_{2m+1}^{(2n)} - C_{2m+3}^{(2n)} \tag{2.15}$$

where $C_{2m+1}^{(2n)}$ counts the number of paths of length $2n$ on the half-line, starting from the origin ($h(0)=1$) and terminating at height $2m+1$ ($h(2n)=2m+1$), easily computed as

$$C_{2m+1}^{(2n)} = \binom{2n}{n-m} - \binom{2n}{n-m-1} \tag{2.16}$$

and in particular $C_1^{(2n)} = c_n$ of (2.1).

$m\backslash n$	1	2	3	4	5	6	7	8	9	10
1	1	2	4	8	15	22	0	-208	-1326	-6460
2		1	4	13	40	121	364	1092	3264	9690
3			1	6	26	100	364	1288	4488	15504
4				1	8	43	196	820	3264	12597
5					1	10	64	336	1581	6954
6						1	12	89	528	2755
7							1	14	118	780
8								1	16	151
9									1	18
10										1

Table II: The powers $a_{m,n}^{(2)}$ of U_m in the meander determinant of order $2n$, $\Delta_{2n}^{(2)}(\beta)$, for $n = 1, 2, ..., 10$.

2.4. Generalizations

The remainder of this paper consists of various generalizations of this determinant formula. The above discussion is strongly related to the $sl(2)$ Lie algebra. Apart from the fact that we considered paths on the Weyl chamber of $sl(2)$ (the half-line), the Temperley-Lieb algebra is known to be a certain quotient of the Hecke algebra $H_n(\beta)$. The latter is defined by generators 1, e_1, e_2,... e_{n-1} and relations

$$\begin{aligned} e_i^2 &= \beta e_i \quad \text{for } i = 1, 2, ..., n-1 \\ e_i e_j &= e_j e_i \quad \text{for } |i-j| > 1 \\ e_i e_{i+1} e_i - e_i &= e_{i+1} e_i e_{i+1} - e_{i+1} \quad \text{for } i = 1, 2, .., n-2 \end{aligned} \tag{2.17}$$

This algebra is usually defined through the generators

$$T_i = q^{1/2}(q^{1/2} - e_i) \tag{2.18}$$

where $\beta = q^{1/2} + q^{-1/2}$, as a deformation of the symmetric group algebra (in particular, the three-term relation reads simply $T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}$). In terms of these latter generators, the quantites $e_i e_{i+1} e_i - e_i$, by which we have to quotient the algebra to recover $TL_n(\beta)$ (see (2.3)), are simply the generalized Young antisymmetrizers of order 3, namely

$$A(T_i, T_{i+1}) = 1 - q^{-1}T_i - q^{-1}T_{i+1} + q^{-2}T_i T_{i+1} + q^{-2}T_{i+1}T_i - q^{-3}T_i T_{i+1} T_i \tag{2.19}$$

easily reexpressed in terms of the e_i's as

$$Y(e_i, e_{i+1}) = q^{3/2} A(T_i, T_{i+1}) = e_i e_{i+1} e_i - e_i \tag{2.20}$$

Requiring the vanishing of (2.19) bears a strong analogy with the $SU(2)$ representations (allowing only for Young tableaux with at most two lines), which can actually be made very precise, and we will return to it in later sections[3].

[3] The special $SU(N)$ quotients of the Hecke algebra we will consider are also known as the commutants of the quantum enveloping algebras $U_q(sl(N))$ [12], and appear in the definition of the A_{N-1} RSOS models of [13].

For the moment, we will content ourselves with the natural generalizations (to $SU(N)$) of the Temperley-Lieb algebra by performing quotients of the Hecke algebra by the generalized Young antisymmetrizer of order $N+1$, $A(T_1, T_2, ..., T_N) \equiv A(e_1, ..., e_N)$

$$A(e_1, e_2, ..., e_N) \;=\; \sum_{w \in S_{N+1}} (-q)^{-l(w)} T_w \tag{2.21}$$

and its shifted versions under $e_j \to e_{j+i-1}$, for $j = 1, 2, ..., N-1$. In (2.21), the sum extends over all the permutations of $N+1$ objects, $l(w)$ is the length of the permutation (the number of factors in any minimal expression of w as a product over transpositions of neighbors $(i, i+1)$), and $T_w = T_{i_1} T_{i_2} ... T_{i_{l(w)}}$ if $w = (i_1, i_1+1)(i_2, i_2+1)...(i_{l(w)}, i_{l(w)}+1)$ (note that this expression is independent of the particular minimal decomposition of w, thanks to the relation $T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}$). We will denote by $H_n^{(N)}(\beta)$ the corresponding $SU(N)$ quotient of the Hecke algebra. In particular, we have $TL_n(\beta) = H_n^{(2)}(\beta)$.

In terms of the Murphy operators [14] [16], defined as

$$L_m = q^{-1} T_{m-1} + q^{-2} T_{m-2} T_{m-1} T_{m-2} + ... + q^{-m+1} T_1 T_2 ... T_{m-2} T_{m-1} T_{m-2} ... T_2 T_1 \tag{2.22}$$

for $m \geq 2$, $L_1 = 0$, it is possible to write compact expressions for the Young antisymmetrizers of order N:

$$A(e_1, e_2, ..., e_{N-1}) \;=\; \prod_{m=2}^{N} (1 - L_m) \tag{2.23}$$

In the following, we will use the various following versions of the Young antisymmetrizer of order N, which are all proportional to A (2.21):

$$\begin{aligned} y(e_1, ..., e_{N-1}) \;&=\; \prod_{m=2}^{N} \frac{1 - L_m}{1 + q^{-1} + ... + q^{-m+1}} \\ E(e_1, ..., e_{N-1}) \;&=\; \prod_{m=2}^{N} q^{\frac{m-1}{2}} (1 - L_m) \end{aligned} \tag{2.24}$$

The antisymmetrizer $y(e_1, ..., e_{N-1})$ is idempotent, $y^2 = y$. As mentioned before, the argument $(e_1, ..., e_{N-1})$ of A, y, E may be shifted into $(e_i, ..., e_{i+N-2})$, and the corresponding functions may be expressed through analogous products, by performing the same shifts in L_m. Finally, we will also use the following version of the Young antisymmetrizer, which has the advantage of being simply expressed in terms of the e_i's, through a recursion (see [10]), starting with $Y(e_i) = e_i$, and

$$Y(e_i, e_{i+1}, ..., e_{i+p}) \;=\; Y(e_i, ..., e_{i+p-1})(e_{i+p} - \mu_p) Y(e_i, ..., e_{i+p-1}) \tag{2.25}$$

for all $i, p \geq 1$, where we have introduced the quantities

$$\mu_p \equiv \mu_p(\beta) = \frac{U_{p-1}(\beta)}{U_p(\beta)} \tag{2.26}$$

in terms of the Chebyshev polynomials (2.14), for all $p \geq 1$. In particular, we have

$$Y(e_i, e_{i+1}) = e_i(e_{i+1} - \mu_1)e_i = e_i e_{i+1} e_i - e_i \tag{2.27}$$

as $\mu_1 = \beta^{-1}$ and $e_i^2 = \beta e_i$. The three antisymmetrizers y, E, Y are proportional to A. In particular we have

$$\begin{aligned} y(e_i, ..., e_{i+N-2}) &= \alpha_N E(e_i, ..., e_{i+N-2}) \\ y(e_i, ..., e_{i+N-2}) &= \gamma_N Y(e_i, ..., e_{i+N-2}) \end{aligned} \tag{2.28}$$

where we have introduced the proportionality constants

$$\begin{aligned} \alpha_N &= \prod_{i=1}^{N-1} (\mu_i)^{N-i} \\ \gamma_N &= \prod_{i=1}^{N-1} (\mu_i)^{2^{N-i-1}} \end{aligned} \tag{2.29}$$

with $\alpha_{N+1}/\alpha_N = \mu_1\mu_2...\mu_N$, and $\gamma_{N+1}/(\gamma_N)^2 = \mu_N$. The second relation of (2.28) is proved by induction on N, by first showing that

$$(1 - L_{N+1})(e_N - q^{-1/2}) = (e_N - q^{1/2})(1 - L_N) + q^{-1/2} \tag{2.30}$$

(also valid for any shift of the e's), and finally deducing that

$$y(e_i, ..., e_{i+N-1}) = \mu_N y(e_i, ..., e_{i+N-2})(e_{i+N-1} - \mu_{N-1})y(e_i, ..., e_{i+N-2}) \tag{2.31}$$

As y is idempotent, we also have the relation

$$Y(e_i, ..., e_{i+N-2})^2 = \gamma_N^{-1} Y(e_i, ..., e_{i+N-2}) \tag{2.32}$$

In the following, we suggest a generalization of meanders into pairs of $SU(N)$ walk diagrams (see definitions below), and the meander determinant will be generalized into the Gram determinant of the basis of some ideal of the $SU(N)$ quotient $H_{Nn}^{(N)}(\beta)$ of the Hecke algebra, the basis elements being in one-to-one correspondence with $SU(N)$ walk diagrams. For the sake of simplicity, we will start with a detailed study of the $SU(3)$ meanders, before going to the general N case.

3. SU(3) meander determinant

In this section, we generalize the concept of meander to $SU(3)$ through the walk diagram picture. A generalized meander is a couple of closed paths (or walk diagrams) starting and ending at the origin of the Weyl chamber for the $sl(3)$ Lie algebra. The bilinear form is provided by the standard scalar product of the Hecke algebra. The $SU(3)$ meander determinant is obtained by an explicit Gram-Schmidt orthogonalization of the walk-diagram basis of a certain ideal of the $SU(3)$ quotient of the Hecke algebra.

3.1. SU(3) walk diagrams

Let us denote by $\Lambda = (\lambda_1, \lambda_2)$ the elements of the weight lattice P of the $sl(3)$ Lie algebra namely the linear combinations $\Lambda = \lambda_1\omega_1 + \lambda_2\omega_2$, $\lambda_1, \lambda_2 \in \mathbb{Z}$, of the two fundamental weights ω_1, ω_2, with $\omega_1^2 = \omega_2^2 = 2/3$ and $\omega_1 \cdot \omega_2 = 1/3$. The Weyl chamber P_+ is the quotient of the weight lattice by the Weyl group, generated by the reflections w.r.t. the walls $\lambda_1 = 0$ and $\lambda_2 = 0$. A representative is given by

$$P_+ \;=\; \{(\lambda_1, \lambda_2) \in P \;\text{ such that } \lambda_1, \lambda_2 \geq 1\} \tag{3.1}$$

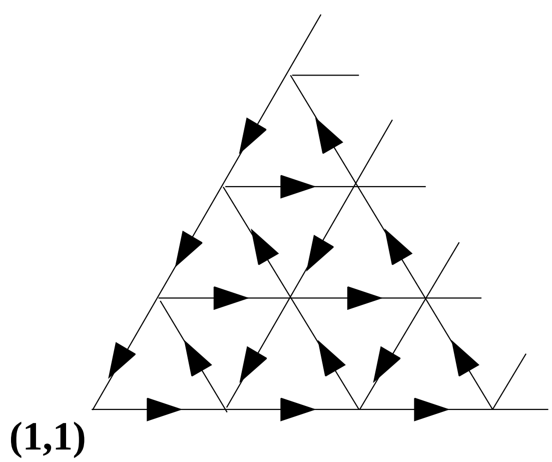


Fig. 2: The simplex Π_+. The three oriented links correspond respectively to ϵ_1 (right), ϵ_2 (up, left) and ϵ_3 (down, left). We have also indicated the origin $(1,1)$.

The Weyl chamber is made into a simplex Π_+ by adding three types of oriented bonds linking the weights (see Fig.2), along the vectors

$$\epsilon_1 \ = \ \omega_1 \qquad \epsilon_2 \ = \ \omega_2 - \omega_1 \qquad \epsilon_3 = -\omega_2 \tag{3.2}$$

subject to the relation $\epsilon_1 + \epsilon_2 + \epsilon_3 = 0$. Analogously, P can be made into a simplex Π by the same procedure. We define the origin of P_+ to be the apex $(1,1)$.

A $SU(3)$ walk diagram of order $3n$ is an oriented closed path of length $3n$ on Π_+, starting and ending at the origin. It is uniquely determined by either of the following data

(i) The sequence of its $3n+1$ "weights" in P_+: $\Lambda_0 = (1,1)$, Λ_1, ..., Λ_{3n-1}, $\Lambda_{3n} = (1,1)$, such that $\Lambda_{i+1} - \Lambda_i \in \{\epsilon_1, \epsilon_2, \epsilon_3\}$ for $i = 0, 1, 2, ..., 3n-1$. The index i is referred to as the position of the weight Λ_i in the sequence.

(ii) The sequence of its $3n$ "step" vectors: $v_1 = \epsilon_1$, v_2, ..., v_{3n-1}, $v_{3n} = \epsilon_3$ with $v_i \in \{\epsilon_1, \epsilon_2, \epsilon_3\}$ and $(1,1)+v_1+v_2+...+v_i \in P_+$ for all $i = 1, 2, ..., 3n-1$, and $v_1+...+v_{3n} = 0$.

The two representations are equivalent, as the steps v_i can be interpreted as $v_i = \Lambda_i - \Lambda_{i-1}$ in the sequence of weights of the walk.

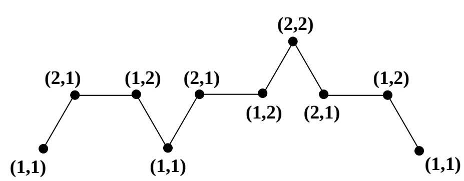


Fig. 3: A sample walk diagram of order 9. We have indicated by dots the successive weights visited by the path.

It will be useful to have a two-dimensional pictorial representation of $SU(3)$ walk diagrams, in the same spirit as for the SU(2) walk diagrams of the previous section (see Fig.1). We choose to represent the three possible directions taken from each weight by three different links, with the following correspondence:

$$\varepsilon_1 \equiv \ / \qquad \varepsilon_2 \equiv \ \text{——} \qquad \varepsilon_3 \equiv \ \backslash \tag{3.3}$$

The walk diagrams are then represented as the corresponding succession of these links, say from left to right (see the example of Fig.3). We denote by $W_{3n}^{(3)}$ the set of $SU(3)$ walk diagrams of order $3n$.

As a simple exercise, let us count the number $c_{3n}^{(3)}$ of $SU(3)$ walk diagrams of given order $3n$. To do that, it is instructive to first count the number $D^{(N)}_{(\lambda_1,\lambda_2),(\mu_1,\mu_2)}$ of paths of length N *on* Π, starting at (λ_1,λ_2) and ending at (μ_1,μ_2). As we are dealing with paths on Π, there is no restriction other than that each step has to be taken among $\epsilon_1,\epsilon_2,\epsilon_3$. Suppose we are taking a total of p steps ϵ_1, q steps ϵ_2 and r steps ϵ_3, then we must have

$$\nu_1=\mu_1-\lambda_1=p-q\ ,\ \nu_2=\mu_2-\lambda_2=q-r\ ,p+q+r=N \tag{3.4}$$

hence

$$p=\frac{N+2\nu_1+\nu_2}{3}\ ,\ q=\frac{N-\nu_1+\nu_2}{3}\ ,r=\frac{N-\nu_1-2\nu_2}{3} \tag{3.5}$$

only valid for $N+2\nu_1+\nu_2=0$ mod 3 (there is no such path otherwise). The number of paths is therefore equal to the number of choices of these p,q,r vectors among N, hence

$$D^{(N)}_{(\lambda_1,\lambda_2),(\mu_1,\mu_2)}\ =\ \frac{N!}{\left(\frac{N+2\nu_1+\nu_2}{3}\right)!\left(\frac{N-\nu_1+\nu_2}{3}\right)!\left(\frac{N-\nu_1-2\nu_2}{3}\right)!} \tag{3.6}$$

where $\nu_1=\mu_1-\lambda_1$, $\nu_2=\mu_2-\lambda_2$. Note that by translational invariance of Π

$$D^{(N)}_{(\lambda_1,\lambda_2),(\mu_1,\mu_2)}\ =\ D^{(N)}_{(0,0),(\nu_1,\nu_2)}\ \equiv\ D^{(N)}_{(\nu_1,\nu_2)} \tag{3.7}$$

where we drop the origin $(0,0)$ in the last shorthand notation. Let us now compare the paths of length $N=3n$, from $(1,1)$ to itself, on Π_+ and on Π. On the latter simplex, the paths can freely cross the walls of the Weyl chamber, hence there are many more of them than on Π_+. But the latter are obtained by reflecting any path on Π w.r.t. the walls of the Weyl chamber, in order to bring it back in P_+. Multiple reflections may be needed to achieve this. This will eventually lead to a surjective map from the paths on Π to those on Π_+. To enumerate the $c_{3n}^{(3)}$ paths on Π_+, we have to start from those on Π, then subtract those which cross the walls of the Weyl chamber. Denoting by s_1 and s_2 the reflections w.r.t. the walls $\lambda_2=0$ and $\lambda_1=0$, we have

$$\begin{aligned} s_1(\lambda_1,\lambda_2)\ &=\ (\lambda_1+\lambda_2,-\lambda_1)\\ s_2(\lambda_1,\lambda_2)\ &=\ (-\lambda_2,\lambda_1+\lambda_2)\\ s_2s_1(\lambda_1,\lambda_2)\ &=\ (\lambda_1,-\lambda_1-\lambda_2)\\ s_1s_2(\lambda_1,\lambda_2)\ &=\ (-\lambda_1-\lambda_2,\lambda_2)\\ s_1s_2s_1(\lambda_1,\lambda_2)\ &=\ (-\lambda_2,-\lambda_1) \end{aligned} \tag{3.8}$$

which together with the identity form the six elements of the Weyl group of $sl(3)$ (identified with the permutation group of three objects S_3). Hence the six possible reflections of the origin $(1,1)$ read

$$(1,1) \quad (-1,2) \quad (2,-1) \quad (1,-2) \quad (-2,1) \quad (-1,-1) \tag{3.9}$$

The correct subtraction formula reads

$$\begin{aligned} c_{3n}^{(3)} &= \sum_{\sigma \in S_3} (-1)^{l(\sigma)} D_{\sigma(1,1),(1,1)}^{(3n)} \\ &= D_{(0,0)}^{(3n)} - D_{(2,-1)}^{(3n)} - D_{(-1,2)}^{(3n)} + D_{(3,0)}^{(3n)} + D_{(0,3)}^{(3n)} - D_{(2,2)}^{(3n)} \\ &= 2\frac{(3n)!}{(n+2)!(n+1)!n!} \end{aligned} \tag{3.10}$$

where the alternate sum ($l(\sigma)$ is the length of the permutation σ, counting the number of reflections w.r.t. walls) accounts for the subtraction of all the paths crossing the walls $\lambda_2 = 0$ and $\lambda_1 = 0$, avoiding oversubtracting. The formula for $c_{3n}^{(3)}$ is a direct generalization of that for the Catalan numbers (2.1) which count the number of $SU(2)$ walk diagrams of order $2n$. The first few numbers $c_{3n}^{(3)}$ are listed in Table III.

n	1	2	3	4	5	6	7	8
$c_{3n}^{(3)}$	1	5	42	462	6006	87516	1385670	23371634

Table III: The numbers $c_{3n}^{(3)}$ of $SU(3)$ walk diagrams of order $3n$, for $n = 1, 2, ..., 8$.

For later use, let us also derive a formula for the numbers $C_{(\lambda_1,\lambda_2)}^{(N)}$ of paths of N steps on Π_+, starting at the origin $(1,1)$ and ending at the weight $(\lambda_1, \lambda_2) \in P_+$. It is clear that $C_{(\lambda_1,\lambda_2)}^{(N)}$ vanishes unless $N + 2\lambda_1 + \lambda_2 = 0$ mod 3. The computation is strictly analogous to that of $c_{3n}^{(3)} = C_{(1,1)}^{(3n)}$: we must subtract from the corresponding paths on Π, $D_{(1,1),(\lambda_1,\lambda_2)}^{(N)} = D_{(\lambda_1-1,\lambda_2-1)}^{(N)}$, the ones which cross the walls of the Weyl chamber, resulting in

$$\begin{aligned} C_{(\lambda_1,\lambda_2)}^{(N)} &= \sum_{\sigma \in S_3} (-1)^{l(\sigma)} D_{\sigma(1,1),(\lambda_1,\lambda_2)}^{(N)} \\ &= D_{(\lambda_1-1,\lambda_2-1)}^{(N)} - D_{(\lambda_1+1,\lambda_2-2)}^{(N)} - D_{(\lambda_1-2,\lambda_2+1)}^{(N)} \\ &+ D_{\lambda_1+3,\lambda_2)}^{(N)} + D_{(\lambda_1,\lambda_2+3)}^{(N)} - D_{(\lambda_1+1,\lambda_2+1)}^{(N)} \\ &= \frac{\lambda_1\lambda_2(\lambda_1+\lambda_2)N!}{\left(\frac{N+2\lambda_1+\lambda_2}{3}+1\right)!\left(\frac{N-\lambda_1+\lambda_2}{3}+1\right)!\left(\frac{N-\lambda_1-2\lambda_2}{3}+1\right)!} \end{aligned} \tag{3.11}$$

3.2. SU(3) quotient and ideal of the Hecke algebra

As mentioned above, we will now concentrate on the $SU(3)$ quotient of the Hecke algebra, obtained by adding to the relations (2.17) the vanishing of all Young antisymmetrizers of order 4, which take the simple form

$$Y(e_i, e_{i+1}, e_{i+2}) \;=\; Y(e_i, e_{i+1})(e_{i+2} - \mu_2)Y(e_i, e_{i+1}) \;=\; 0 \tag{3.12}$$

for $i = 1, 2, ..., n-3$. Noting that

$$Y(e_i, e_{i+1})^2 \;=\; (\mu_1^2 \mu_2)^{-1} Y(e_i, e_{i+1}) \tag{3.13}$$

(see (2.32)), the vanishing of (3.12) translates into

$$(e_i e_{i+1} e_{i+2} - e_{i+2} - e_i)Y(e_i, e_{i+1}) \;=\; 0 \tag{3.14}$$

for all $i = 1, 2, ..., n-3$.

The notion of reduced element has to be slightly generalized for $H_n^{(3)}(\beta)$ and the higher Hecke quotients. Indeed, the relations (2.17) and (3.14) can be used repeatedly to reduce any element of $H_n^{(3)}(\beta)$ to a linear combination of "reduced elements", which take the form of products of e_i's with the smallest possible number of factors. However, if we try to enumerate these reduced elements, we find non-trivial vanishing linear combinations between them. For instance, due to (2.17), we have $e_i e_{i+1} e_i - e_i + e_{i+1} - e_{i+1} e_i e_{i+1} = 0$. It turns out that the notion of reduced element is better (and usually) defined in terms of the generators $T_i = q^{1/2}(q^{1/2} - e_i)$ mentioned above, thanks to the relation $T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}$, as the products of T_i's with the smallest numbers of factors. This alternative description replaces the above unwanted linear combinations by identities between various reduced elements, which can therefore be easily enumerated. However, in view of the $SU(2)$ case, we must insist here on working with the e_i's instead of the T_i's, and we will construct a basis of $H_n^{(3)}(\beta)$ made only of reduced elements in the e_i's (this will be done in all generality in Sect.5).

Our immediate task however is not to construct a general basis of $H_n^{(3)}(\beta)$ but rather of a particular ideal of $H_{3n}^{(3)}(\beta)$. By analogy with the $SU(2)$ case, let us consider the left ideal $\mathcal{I}_{3n}^{(3)}(\beta)$ of $H_{3n}^{(3)}(\beta)$ generated by the element

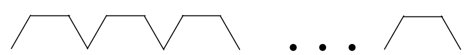
$$Y_{3n}^{(3)} \;=\; Y(e_1, e_2)Y(e_4, e_5)...Y(e_{3n-2}, e_{3n-1}) \tag{3.15}$$

Let us now construct a basis of reduced elements of this ideal using the $SU(3)$ walk diagrams of order $3n$. By reduced element we mean here a product of e_i's times $Y_{3n}^{(3)}$, with the smallest number of factors.

Like in the $SU(2)$ case, let us first reexpress the walk diagrams of $W_{3n}^{(3)}$ in terms of box additions. We start from the fundamental $SU(3)$ walk diagram $a_0^{(3)}$, with weights

$$
\begin{aligned}
\Lambda_0 &= (1,1) = \Lambda_3 = ... = \Lambda_{3n} \\
\Lambda_1 &= (2,1) = \Lambda_4 = ... = \Lambda_{3n-2} \\
\Lambda_2 &= (1,2) = \Lambda_5 = ... = \Lambda_{3n-1}
\end{aligned}
\tag{3.16}
$$

This is the most compact path of length $3n$ on Π_+. In the abovementioned pictorial representation $a_0^{(3)}$ reads

$a_0^{(3)} =$ 

(3.17)

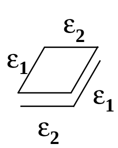
For a general walk $a \in W_{3n}^{(3)}$, we define three types of box additions at position i, still denoted by $a \to a + \diamond_i$, according to the configuration of the weights Λ_{i-1}, Λ_i, Λ_{i+1} of a at positions $i-1$, i and $i+1$:

(i) $\Lambda_{i+1} - \Lambda_i = \epsilon_1$ and $\Lambda_i - \Lambda_{i-1} = \epsilon_2$. A box addition at position i transforms $\Lambda_i \to \Lambda_i + \epsilon_1 - \epsilon_2$ and leaves all the other weights unchanged.

(ii) $\Lambda_{i+1} - \Lambda_i = \epsilon_1$ and $\Lambda_i - \Lambda_{i-1} = \epsilon_3$. A box addition at position i transforms $\Lambda_i \to \Lambda_i + \epsilon_1 - \epsilon_3$ and leaves all the other weights unchanged.

(iii) $\Lambda_{i+1} - \Lambda_i = \epsilon_2$ and $\Lambda_i - \Lambda_{i-1} = \epsilon_3$. A box addition at position i transforms $\Lambda_i \to \Lambda_i + \epsilon_2 - \epsilon_3$ and leaves all the other weights unchanged.

Pictorially, this is summarized by the following box additions, according to the case at hand.

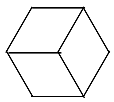
(i) : 

(ii) :

(3.18)

If the weights of a are not in one of the three cases $(i)-(iii)$ above, the box addition cannot be performed at the position i. For instance, on the fundamental walk $a_0^{(3)}$, box additions can be performed only at positions $3, 6, 9, ..., 3n-3$, and fall in the case (ii). This construction gives a procedure to describe any $SU(3)$ walk diagram as a sequence of box additions on the fundamental walk $a_0^{(3)}$. This description is however not unique,

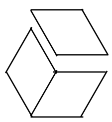
as different sequences may lead to the same walk diagram. The order in which the box additions are made is not a problem, the only difficulty here is the occurence of hexagons in the box decomposition of a (i.e. the filling of the space between $a_0^{(3)}$ and a with boxes of type $(i)-(iii)$), because there are two different ways of filling an hexagon with boxes, namely



or

(3.19)

To fix this ambiguity, we simply forbide any box addition on a which would create an hexagon of the second type in (3.19), namely we do not allow the following box addition at position i



(3.20)

With this latter rule, each walk diagram $a \in W_{3n}^{(3)}$ has a unique box decomposition, namely a non-ordered sequence of box additions to be performed on $a_0^{(3)}$ leading to a. Such a box decomposition can be pictorially represented by filling the space between $a_0^{(3)}$ and a with the corresponding boxes.

We are now ready to establish a map φ between $W_{3n}^{(3)}$ and the reduced elements of $\mathcal{I}_{3n}^{(3)}(\beta)$. We start with the fundamental walk

$$\varphi(a_0^{(3)}) \;=\; Y_{3n}^{(3)} \tag{3.21}$$

defined in (3.15), and construct all the other reduced elements by induction on box additions, namely

$$\varphi(a+\diamond_i) \;=\; e_i\varphi(a) \tag{3.22}$$

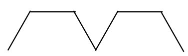
for all $a \in W_{3n}^{(3)}$. This expression is well-defined, as at each step, the various box additions which can be performed say at positions i and j on a diagram a satisfy $|j-i|>1$, hence the corresponding e_i and e_j commute: the order of their left multiplication does not matter. Moreover, we have taken care of the hexagon ambiguities[4] by forbiding (3.20). This leads to the definition of the basis 1 of $\mathcal{I}_{3n}^{(3)}(\beta)$, with elements

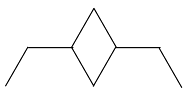
$$(a)_1 \;=\; (\mu_1^2\mu_2)^{\frac{n}{2}}\;\varphi(a)\;,\; a \in W_{3n}^{(3)} \tag{3.23}$$

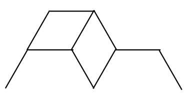
[4] Having forbidden all the hexagons (3.20), the only hexagons appearing in the box decomposition of any $a \in W_{3n}^{(3)}$ are of the form of the first hexagon of (3.19). We could have decided to

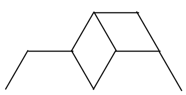
(The choice of the normalization factor will become clear below.). As an immediate consequence the vector space $\mathcal{I}_{3n}^{(3)}(\beta)$ has dimension[5] $c_{3n}^{(3)}$ (3.10).

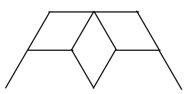
Let us illustrate this construction with the case $n = 2$. There are $c_6^{(3)} = 5$ walk diagrams, and the basis 1 of $\mathcal{I}_6^{(3)}(\beta)$ reads

$\Big($  $\Big)_1 = \mu_1^2\mu_2$

$\Big($  $\Big)_1 = \mu_1^2\mu_2$

$\Big($  $\Big)_1 = \mu_1^2\mu_2$

$\Big($  $\Big)_1 = \mu_1^2\mu_2$

$\Big($  $\Big)_1 = \mu_1^2\mu_2$

(3.24)

where we have represented, for each walk diagram, the box additions performed on the fundamental one (the box decompositions). It is instructive to recover the basis (3.24) by a direct study of the left ideal $\mathcal{I}_6^{(3)}(\beta) = H_6^{(3)}(\beta)Y_6^{(3)}$, with $Y_6^{(3)}$ as in (3.15). Noting that

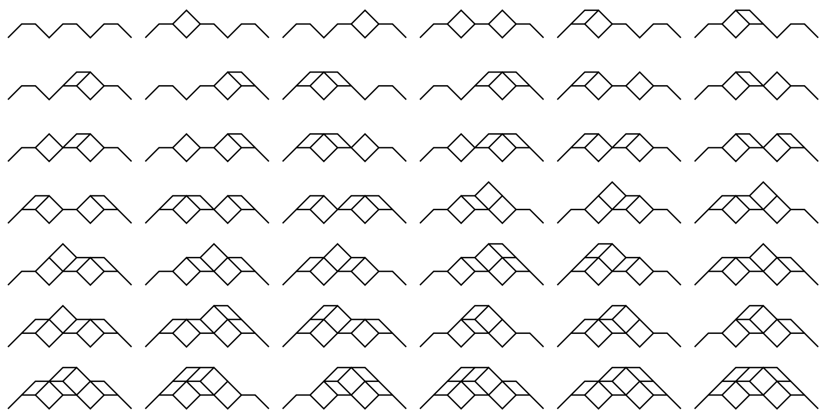
$$e_i Y(e_i, e_{i+1}) = e_{i+1} Y(e_i, e_{i+1}) = \beta Y(e_i, e_{i+1}) \tag{3.25}$$

we see that the only new element of $\mathcal{I}_6^{(3)}(\beta)$ obtained by acting with one e_i on the fundamental one $Y_6^{(3)}$ is $e_3 Y_6^{(3)}$ confirming the fact that the only possible box addition on $a_0^{(3)}$ here is at position $i = 3$. Acting with two extra e_i's, we easily find the only other elements $e_2 e_3 Y_6^{(3)}$, $e_4 e_3 Y_6^{(3)}$ and $e_2 e_4 e_3 Y_6^{(3)}$. This exhausts all reduced elements of $\mathcal{I}_6^{(3)}(\beta)$, as $e_3 e_2 e_4 e_3 Y_6^{(3)} = (\beta e_3 + (e_2 + e_4)e_3 - 2)Y_6^{(3)}$. Note that we still have not met here any

make a more symmetric choice for φ, namely by associating the combination $Y(e_i, e_{i+1})$ instead of $e_i e_{i+1} e_i$ to each of these hexagons (which would then be represented empty, without their inner box decomposition). This however would not affect the final value of the meander determinant, allowing us to stick to our non-symmetric choice. The symmetric choice would have the only advantage of putting the hexagons in the box decomposition on the same footing as those over which the walk rests (i.e., forming the product $Y_{3n}^{(3)}$, defining the ideal).

[5] Let us stress that this basis is distinct from the standard basis of [14] [16], when restricted to the ideal $\mathcal{I}_{3n}^{(3)}(\beta)$. The latter uses indeed the generators T_i (2.18). Our non-standard choice finds its justification in the $SU(2)$ case, in which meanders are recovered.

hexagon ambiguity, occurring only for $n \geq 3$. For completeness, we list below the box decompositions relevant to the $n = 3$ case, with $c_9^{(3)} = 42$ walk diagrams:



(3.26)

Note that the third diagram in the first line of (3.26) is nothing but the box decomposition of the sample walk of Fig.3. Note also that only hexagons of the first type of (3.19) appear in the above box decompositions.

A $SU(3)$ meander is now identified as a pair $(a)_1, (b)_1$ of basis 1 elements for $\mathcal{I}_{3n}^{(3)}(\beta)$. Following the $SU(2)$ example, we may attach to each meander the quantity $\big((a)_1, (b)_1\big) = \mathrm{Tr}\big((a)_1 (b)_1^t\big)$. Here, the Markov trace on $H_{3n}^{(3)}$ (still denoted by Tr) is normalized so that

$$\mathrm{Tr}(1) \;=\; U_2(\beta)^{3n} \;=\; (\beta^2 - 1)^{3n} \tag{3.27}$$

and still defined by induction through the relation (2.7), but with a different constant η, namely

$$\eta \;=\; \mu_2 \;=\; \frac{\beta}{\beta^2 - 1} \tag{3.28}$$

Let us consider the Gram matrix of the basis 1, with entries

$$\big[\mathcal{G}_{3n}^{(3)}(\beta)\big]_{a,b} \;=\; \big((a)_1, (b)_1\big) \tag{3.29}$$

As an example, the Gram matrix for $n = 2$ reads (with the same ordering of the basis elements as in (3.24))

$$\mathcal{G}_6^{(3)}(\beta) \;=\; \beta^2(\beta^2 - 1) \begin{pmatrix} \beta^2 - 1 & \beta & \beta^2 & \beta^2 & \beta^3 \\ \beta & \beta^2 & 2\beta & 2\beta & \beta^2 + 2 \\ \beta^2 & 2\beta & 2\beta^2 & \beta^2 + 2 & \beta(\beta^2 + 2) \\ \beta^2 & 2\beta & \beta^2 + 2 & 2\beta^2 & \beta(\beta^2 + 2) \\ \beta^3 & \beta^2 + 2 & \beta(\beta^2 + 2) & \beta(\beta^2 + 2) & \beta^2(\beta^2 + 2) \end{pmatrix} \tag{3.30}$$

3.3. $SU(3)$ meander determinant: main result

We define the $SU(3)$ meander determiant $\Delta^{(3)}_{3n}(\beta)$ as the determinant of the Gram matrix (3.29) of the basis 1 of $\mathcal{I}^{(3)}_{3n}(\beta)$. The aim of this section is to prove the following formula for this determinant

$$\Delta^{(3)}_{3n}(\beta) \;=\; \prod_{m=1}^{n+1} \big[U_m(\beta)\big]^{a^{(3)}_{m,n}} \tag{3.31}$$

where

$$\begin{aligned} a^{(3)}_{m,n} \;&=\; \sum_{\sigma\in S_3} (-1)^{l(\sigma)} C^{(3n)}_{(m+1,m+1)-\sigma(1,1)} \\ &=\; C^{(3n)}_{(m,m)} - C^{(3n)}_{(m+2,m-1)} - C^{(3n)}_{(m-1,m+2)} \\ &+ C^{(3n)}_{(m+3,m)} + C^{(3n)}_{(m,m+3)} - C^{(3n)}_{(m+2,m+2)} \end{aligned} \tag{3.32}$$

for $m \geq 2$ and

$$a^{(3)}_{1,n} \;=\; C^{(3n)}_{(4,1)} + C^{(3n)}_{(1,4)} - C^{(3n)}_{(3,3)} \tag{3.33}$$

in terms of the numbers $C^{(N)}_{(\lambda_1,\lambda_2)}$ of paths of length N on Π_+ starting from the origin $(1,1)$ and terminating at (λ_1,λ_2), computed in (3.11) (it is understood that $C^{(3n)}_{(\lambda_1,\lambda_2)}$ vanishes unless $(\lambda_1,\lambda_2)\in P_+$). The first few values of the powers $a^{(3)}_{m,n}$ of U_m in the $SU(3)$ meander determinant $\Delta^{(3)}_{3n}(\beta)$ are listed in Table IV.

$m\backslash n$	1	2	3	4	5	6	7	8
1	1	6	42	297	1430	-14586	-764218	-21246940
2	1	6	63	814	11583	175032	2762942	45108888
3		4	42	506	7306	119340	2098208	38571368
4			21	374	5707	89352	1495490	26803832
5				121	3276	65790	1218356	22309287
6					728	27336	701879	15622750
7						4488	218994	6931694
8							28101	1701678
9								177859

Table IV: The powers $a^{(3)}_{m,n}$ of U_m in the $SU(3)$ meander determinant of order $3n$, $\Delta^{(3)}_{3n}(\beta)$, for $n=1,2,...,8$.

The formula (3.31) exhibits a remarkable feature: the numbers $a^{(3)}_{m,n}$ are obtained from the C's by the same addition/subtraction formula as that giving the C's in terms of the D's, namely between the numbers of paths on Π_+ and those on Π (see (3.11)). This feature was already present in the $SU(2)$ case, if we note that eqs.(2.13)(2.16) translate into

$$\begin{aligned} C^{(2n)}_{2m+1} \ &= \ D^{(2n)}_{2m} - D^{(2n)}_{2m+2} \\ a^{(2)}_{m,n} \ &= C^{(2n)}_{2m+1} - C^{(2n)}_{2m+3} \end{aligned} \tag{3.34}$$

where we have introduced the numbers $D^{(2n)}_{2m} = \binom{2n}{n-m}$ of paths of $2n$ steps from 0 to $2m$ on the integer line $\mathbb{Z}$, identified with the weight lattice P of $SU(2)$.

The validity of (3.31) is readily checked in the case $n = 2$, where we find (see also Table IV), by direct computation of the determinant of (3.30)

$$\Delta^{(3)}_6(\beta) \ = \ U_1^6 U_2^6 U_3^4 \tag{3.35}$$

in agreement with $C^{(6)}_{(1,1)} = C^{(6)}_{(5,2)} = C^{(6)}_{(1,4)} = 5$, $C^{(6)}_{(4,1)} = 10$, $C^{(6)}_{(2,2)} = 16$, $C^{(6)}_{(2,5)} = 0$ and $C^{(6)}_{(3,3)} = 9$.

The formula (3.31) is proved below, by the explicit Gram-Schmidt orthogonalization of the basis 1, namely the construction of a new basis (which we call basis 2), still indexed by the walk diagrams $a \in W^{(3)}_{3n}$, and such that

$$(a)_2 \ = \ \sum_{\substack{b \in W^{(3)}_{3n} \\ b \subset a}} P_{a,b} (b)_1 \tag{3.36}$$

where the sum extends over the walk diagrams b "included" in a, namely such that a can be obtained from b by box additions. The basis 2 is orthonormal w.r.t. the scalar product $(\ ,\ )$, namely $\big((a)_2, (b)_2\big) = \delta_{a,b}$ for any $a, b \in W^{(3)}_{3n}$.

3.4. The orthonormal basis

The orthonormal basis 2 is constructed as follows. We start with the fundamental element

$$\begin{aligned} (a^{(3)}_0)_2 \ &= \ (\mu_1^2\mu_2)^{n/2} (a^{(3)}_0)_1 \ = \ (\mu_1^2\mu_2)^n Y^{(3)}_{3n} \\ &= \ y(e_1, e_2) y(e_4, e_5) ... y(e_{3n-2}, e_{3n-1}) \end{aligned} \tag{3.37}$$

with $Y^{(3)}_{3n}$ as in (3.15). The last line of (3.37) is a reexpression in terms of the idempotent antisymmetrizers of order 3 of (2.28)(2.29). The normalization factor in (3.37) ensures that

$(a_0^{(3)})_2$ has norm 1, as $\big((a_0^{(3)})_1,(a_0^{(3)})_1\big) = \mathrm{Tr}(Y_{3n}^{(3)}) = (\mu_1^2\mu_2)^{-n}$, by immediate application of the Markov property (2.7). The other basis 2 elements are constructed by box additions on the fundamental one, through the recursion relation

$$(a+\diamond_{i,m})_2 \;=\; \sqrt{\frac{\mu_{m+1}}{\mu_m}}(e_i-\mu_m)\;(a)_2 \tag{3.38}$$

where m is the "height" of the box, defined as

$$m \;=\; \Lambda_i\cdot(\Lambda_{i+1}+\Lambda_{i-1}-2\Lambda_i) \tag{3.39}$$

in terms of the weights Λ_{i-1}, Λ_i, Λ_{i+1} of a with respective positions $i-1,i,i+1$, for all $1\leq i\leq 3n-1$ (note that the effect of the box addition at position i is to change $\Lambda_i\to\Lambda_{i+1}+\Lambda_{i-1}-\Lambda_i$ in all cases (i)-(iii) of (3.18)). Note that as (3.38) depends explicitly on the height m of the box addition, we have added the subscript m to the box symbol $\diamond_{i,m}$. For simplicity, we will denote by $(\diamond_{i,m})$ the element of the Hecke algebra which multiplies $(a)_2$ in (3.38). The fundamental property of (3.38) is that it resolves the hexagon ambiguity, namely the two ways (3.19) of building an hexagon by these box additions are equivalent, i.e.[6],

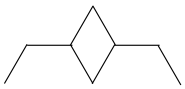
$$(e_i-\mu_m)(e_{i+1}-\mu_{m+p+1})(e_i-\mu_p) \;=\; (e_{i+1}-\mu_p)(e_i-\mu_{m+p+1})(e_{i+1}-\mu_m) \tag{3.40}$$

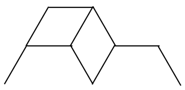
where $m=\Lambda\cdot(\epsilon_1-\epsilon_2)-1$ and $p=\Lambda\cdot(\epsilon_2-\epsilon_3)-1$ and Λ denotes the weight of the leftmost vertex of the hexagons. Eq.(3.40) is easily proved by using the definition (2.26) for μ and the recursion relation for the Chebishev polynomials $U_{m+1}=\beta U_m-U_{m-1}$, together with the Hecke algebra relations (2.17). We also note that, upon defining $\mu_0=0$, the box additions (3.38) enable us to rewrite each of the building blocks $Y(e_i,e_{i+1})$ of the fundamental element $(a_0^{(3)})_2$ as the hexagonal result of three box additions on an empty walk diagram. For each such hexagon, the equivalence (3.40) simply amounts to $Y(e_i,e_{i+1})=Y(e_{i+1},e_i)$. In this way, any basis 2 element can be seen as the result of box additions on the vacuum diagram (identified with the unit 1 of $H_{3n}^{(3)}(\beta)$) as well.

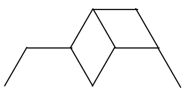
[6] This equation takes the form of the celebrated Yang-Baxter equation for the so-called trigonometric limit of the A_2 RSOS model of [13].

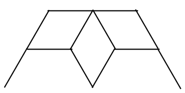
In the example $n = 2$ of (3.24), we have the basis 2 elements

$\Big($  $\Big)_2 = \mu_1^4\mu_2^2$

$\Big($  $\Big)_2 = \mu_1^4\mu_2^3$

$\Big($  $\Big)_2 = \mu_1^{7/2}$

$\Big($  $\Big)_2 = \mu_1^{7/2}$

$\Big($  $\Big)_2 = \mu_1^3\mu_2^5$

(3.41)

The basis 2, determined by (3.37)(3.38), coincides up to normalization factors with the so-called semi-normal basis $(a)_{sn}$ (still indexed by the walk diagrams $a \in W_{3n}^{(3)}$) of [14] [16] when restricted to the ideal $\mathcal{I}_{3n}^{(3)}(\beta)$ (identified with the top Specht module of $H_{3n}^{(3)}(\beta)$). The semi-normal basis elements $(a)_{sn}$ for $\mathcal{I}_{3n}^{(3)}$ satisfy stronger relations, namely that

$$(a)_{sn}^t (b)_{sn} \;=\; 0 \quad \text{unless } a \;=\; b \tag{3.42}$$

Let us assume the analogous relations for the basis 2 elements, and verify that all the $(a)_2$ have norm 1. Let us rewrite the quantity $(a+\diamond_i)_2^t(a+\diamond_i)_2$, by "moving the box" from the left factor to the right: this amounts to an extra left multiplication by $(e_i-\mu_m)^t = (e_i-\mu_m)$ on $(a)_2$, namely

$$\begin{aligned}(a+\diamond_i)_2^t(a+\diamond_i)_2 &= (a)_2^t \frac{\mu_{m+1}}{\mu_m}(e_i-\mu_m)^2(a)_2 \\ &= (a)_2^t \frac{\mu_{m+1}}{\mu_m}\big((\mu_1^{-1}-2\mu_m)(e_i-\mu_m)+\mu_m(\mu_1^{-1}-\mu_m)\big)(a)_2 \\ &= \Big(\frac{1}{\sqrt{\mu_m\mu_{m+1}}}-\sqrt{\mu_m\mu_{m+1}}\Big)(a)_2^t(a+\diamond_i)_2+(a)_2^t(a)_2 \\ &= (a)_2^t(a)_2\end{aligned} \tag{3.43}$$

where we have first used $e_i^2 = \beta e_i = \mu_1^{-1}e_i$, then the recursion relation $\mu_1^{-1}-\mu_m = \mu_{m+1}^{-1}$. We have dropped the term proportional to $(a)_2^t(a+\diamond_i)_2 = 0$ by application of (3.42). Now eq.(3.43), enables us to prove by induction on the box additions that $\big((a)_2,(a)_2\big) = 1$ for all $a \in W_{3n}^{(3)}$, as $\big((a_0^{(3)})_2,(a_0^{(3)})_2\big) = 1$. This fixes the prefactor in (3.38).

The relation (3.42) for basis 2 elements, namely

$$(a)_2^t(b)_2 \;=\; 0 \qquad \text{unless } a \;=\; b \tag{3.44}$$

can be directly proved by induction on the number of boxes, denoted by $|a|$ and $|b|$ in the box decompositions of a and b, in the same spirit as for the $SU(2)$ case (see [11]). Let us give a brief description of this proof for completeness. The aim is to prove by induction on the integer k the following property

$$\mathcal{P}_k \ : \ (a)_2^t(b)_2 \ = \ 0 \quad \text{for any } a \neq b \in W_{3n}^{(3)}, \text{ with } |a| = k \leq |b| \tag{3.45}$$

Assume that $\mathcal{P}_{k-1}$ is true for some $k \geq 1$. Let us consider a pair a, b of walk diagrams with $|a| = k$ and $|b| \geq k$, $b \neq a$. We write the walk a with k boxes as the result of a box addition on some a', at position i, with height ℓ, namely $a = a' + \diamond_{i,\ell}$, and $|a'| = |a| - 1 = k - 1$. We then rewrite the product $(a)_2^t(b)_2 = (a' + \diamond_{i,\ell})_2^t(b)_2$ by letting this box act on $(b)_2$ by left multiplication. Three situations may occur, according to the configuration of the weights Λ_{i-1}, Λ_i, Λ_{i+1} of b at respective positions $i-1$, i, $i+1$. Setting $\Lambda_i - \Lambda_{i-1} = \epsilon_r$ and $\Lambda_{i+1} - \Lambda_i = \epsilon_s$, we have the three possibilities

(i) b has a minimum at position i, namely $(r, s) = (2, 1)$, $(3, 2)$ or $(3, 1)$. Let $m = \Lambda_i \cdot (\epsilon_s - \epsilon_r)$ be the height of the box to be added on b at position i, we simply rewrite

$$(\diamond_{i,\ell}) \ = \ \sqrt{\frac{\mu_{\ell+1}}{\mu_\ell}}(e_i - \mu_\ell) \ = \ \sqrt{\frac{\mu_{\ell+1}\mu_m}{\mu_\ell\mu_{m+1}}}(\diamond_{i,m}) + \sqrt{\frac{\mu_{\ell+1}}{\mu_\ell}}(\mu_m - \mu_\ell) \tag{3.46}$$

hence $(a)_2^t(b)_2$ can be reexpressed as a linear combination of $(a')_2^t(b + \diamond_{i,m})_2$ and $(a')_2^t(b)_2$.

(ii) b has a maximum at position i, namely $(r, s) = (1, 2)$, $(2, 3)$ or $(1, 3)$. Using the equivalence (3.40), we can always arrange for $(b)_2$ to be the result $(b' + \diamond_{i,m})_2$ of a box addition on some b', with $|b'| = |b| - 1$ (we also write $(b')_2 = (b - \diamond_{i,m})_2$). Hence the element $(b')_2$ is multiplied from the left by

$$(\diamond_{i,\ell})(\diamond_{i,m}) \ = \ \sqrt{\frac{\mu_{\ell+1}}{\mu_\ell}}(\mu_{\ell+1}^{-1} - \mu_m)(\diamond_{i,m}) + \sqrt{\frac{\mu_{\ell+1}\mu_m}{\mu_\ell\mu_{m+1}}} \tag{3.47}$$

and $(a)_2^t(b)_2$ is expressed as a linear combination of $(a')_2^t(b)_2$ and $(a')_2^t(b')_2$, with $|b'| = |b| - 1$.

(iii) b has a slope at position i, namely $r = s = 1$, 2 or 3. Without loss of generality, we may assume that b contains the two boxes $f = (\diamond_{i-1,m+1})(\diamond_{i,m})$. The left multiplication by $(\diamond_{i,\ell})$ consists of two terms, one proportional to $e_i f$, and the other proportional to f. The former is proportional to

$$\begin{aligned} e_i(e_{i-1} - \mu_m)(e_i - \mu_{m-1}) \ &= \ Y(e_{i-1}, e_i) - \mu_{m-1} e_i e_{i-1} \\ &= \ \big(e_{i-1}(e_i - \mu_1) - \mu_{m-1} e_i\big) e_{i-1} \end{aligned} \tag{3.48}$$

hence e_i has commuted through the two boxes, creating a left factor of e_{i-1}. Now we can repeat this process, until we meet the bottom of the diagram b. Several cases have to be inspected, let us simply give one example: we are left, say, with the left multiplication of e_j with the bottom boxes $(\diamond_{j-1,1})(\diamond_{j-2,2})Y(e_{j-1},e_j)$, which gives

$$\begin{aligned} e_j(e_{j-1}-\mu_1)(e_{j-2}-\mu_2)Y(e_{j-1},e_j) & \\ &= \mu_1 Y(e_{j-1},e_j)(e_{j-2}-\mu_2)Y(e_{j-1},e_j) \\ &= 0 \end{aligned} \tag{3.49}$$

where we used $Y(e_{j-1},e_j)=\mu_1 e_j Y(e_{j-1},e_j)$, commuted e_j through e_{j-2}, and finally used the vanishing condition of the fourth order antisymmetrizer (3.12) in $H_{3n}^{(3)}(\beta)$.

To summarize, in all cases (i)-(iii) above, we have been able to rewrite $(a)_2^t(b)_2$ as a linear combination of terms of the form $(a')_2^t(b'')_2$, where $b''=b+\diamond_i$, b or $b-\diamond_i$. In all cases, we have $|a'|=|a|-1=k-1$, and $|b''|\geq|b|-1\geq k-1$. Moreover, $b''\neq a'$: otherwise, one would have been necessarily in the case (ii) with $b''=b-\diamond_i=a'=a-\diamond_i$, hence $a=b$, which contradicts the hypothesis. Hence $b''\neq a'$ and we may apply to each pair (a',b'') the induction hypothesis $\mathcal{P}_{k-1}$, hence $(a')_2^t(b'')_2=0$ in all cases at hand, and $\mathcal{P}_k$ follows. There remains to prove $\mathcal{P}_0$. We have $a=a_0^{(3)}$, the only walk with 0 boxes. We simply have to act on the left of $(b)_2$ with the hexagons $Y(e_i,e_{i+1})$ forming $(a)_2^t=(a)_2$. The result vanishes for all $b\neq a_0^{(3)}$, as at least one of these hexagons, say $Y(e_j,e_{j+1})$ has a right factor e_j or e_{j+1}, whose position corresponds to a slope of b (the result of the left multiplication of $(b)_2$ by this yields zero, like in the case (iii) above), or yields directly zero by the vanishing of the antisymmetrizers of order 4. This completes the proof of $\mathcal{P}_k$ for all $k\geq 0$, and (3.44) follows.

As a final remark, the property (3.36) follows directly from the recursive definition (3.38). Indeed, the process of box addition only involves walk diagrams included in a for the construction of $(a)_2$, hence the change of basis $1\to 2$ is triangular.

3.5. $SU(3)$ meander determinant: the proof

Using (3.36), we can easily reexpress the $SU(3)$ meander determinant as

$$\Delta_{3n}^{(3)}(\beta) = \prod_{a\in W_{3n}^{(3)}} P_{a,a}^{-2} \tag{3.50}$$

as the basis 2 is orthonormal, and the change of basis $1 \to 2$ is triangular, with normalization factors $P_{a,a}$ on the diagonal. The quantities $P^2_{a,a}$ are easily computed by induction. First we have, from (3.37)

$$P^2_{a_0^{(3)},a_0^{(3)}} = (\mu_1^2\mu_2)^n \tag{3.51}$$

and from (3.38)

$$P^2_{a+\diamond_{i,m},a+\diamond_{i,m}} = \frac{\mu_{m+1}}{\mu_m}\, P^2_{a,a} \tag{3.52}$$

for all $a \in W^{(3)}_{3n}$. Each term $P^2_{a,a}$ is therefore expressed as a product over all the box additions leading from $a_0^{(3)}$ to a

$$P^2_{a,a} = (\mu_1^2\mu_2)^n \prod_{\substack{\text{box additions} \\ \text{from } a_0^{(3)} \text{ to } a}} \frac{\mu_{m+1}}{\mu_m} \tag{3.53}$$

where m stands for the height of the box addition.

In the $n = 2$ example of (3.24)(3.41), the five walks have respective values of $P^2_{a,a}$

$$\mu_1^4\mu_2^2\ ,\ \mu_1^4\mu_2\mu_3\ ,\ \mu_1^3\mu_2^2\mu_3\ ,\ \mu_1^3\mu_2^2\mu_3\ ,\ \mu_1^2\mu_2^4\mu_3 \tag{3.54}$$

leading immediately to (3.35), using (3.50).

The product in (3.53) can be further simplified, by noting that the powers of μ can be redistributed to each of the individual steps $v_i = \Lambda_i - \Lambda_{i-1}$ forming a. For each such step, say from $\Lambda = (\lambda_1, \lambda_2)$ to $\Lambda' = (\lambda_1', \lambda_2')$, let us define a weight function

$$w(\Lambda, \Lambda') = \begin{cases} \sqrt{\mu_{\lambda_1}\mu_{\lambda_1+\lambda_2}} & \text{if } \Lambda' - \Lambda = \epsilon_1 \\ \sqrt{\mu_{\lambda_2}\mu_{\lambda_1'}} & \text{if } \Lambda' - \Lambda = \epsilon_2 \\ \sqrt{\mu_{\lambda_2'}\mu_{\lambda_1'+\lambda_2'}} & \text{if } \Lambda' - \Lambda = \epsilon_3 \end{cases} \tag{3.55}$$

Now, by inspection of the three possible box additions (3.18), we see that the weights exactly follow the rule (3.38), namely that

$$\prod_{a \in W^{(3)}_{3n}} P^2_{a,a} = \prod_{\substack{\text{steps } v \text{ in all} \\ \text{walks } a \in W^{(3)}_{3n}}} w(v) \tag{3.56}$$

This enables us to identify the total power $\alpha^{(3)}_{m,n}$ of μ_m in the product (3.56). Indeed a factor $\mu_m^{1/2}$ will appear whenever in a step with value ϵ_1, ϵ_2 or ϵ_3 we will have respectively $\lambda_1 = m$ or $\lambda_1 + \lambda_2 = m$, $\lambda_2 = m$ or $\lambda_1' = m$, $\lambda_2' = m$ or $\lambda_1' + \lambda_2' = m$. Counting all

these occurrences involves counting the number of walks $a \in W_{3n}^{(3)}$ which have a fixed step Λ_p, Λ_{p+1}. These paths are made of two pieces:

(i) the portion Λ_0, Λ_1, ..., Λ_p which goes from the origin $\Lambda_0 = (1,1)$ to Λ_p on Π_+. There are $C_{\Lambda_p}^{(p)}$ such paths (see (3.11)).

(ii) the portion Λ_{p+1}, Λ_{p+2}, ..., $\Lambda_{3n} = (1,1)$, which can be thought of as the "reverse" path $\Lambda'_0 = \Lambda^t_{3n} = (1,1)$, $\Lambda'_1 = \Lambda^t_{3n-1}$, ..., $\Lambda'_{3n-p-1} = \Lambda^t_{p+1}$ of $3n-p-1$ steps, from the origin to Λ^t_{p+1} (the superscript t means $(\lambda_1, \lambda_2)^t = (\lambda_2, \lambda_1)$) obtained by "reversing" the directions of all the steps, namely exchange all $\epsilon_1 \leftrightarrow \epsilon_3$. There are $C^{(3n-p-1)}_{\Lambda^t_{p+1}}$ such paths (see (3.11)).

Hence the number of paths with a specified step $(\Lambda_p, \Lambda_{p+1})$ is $C^{(p)}_{\Lambda_p} C^{(3n-p-1)}_{\Lambda^t_{p+1}}$. We are now ready to express the total number of occurrences of μ_m in (3.56), namely

$$\begin{aligned}
\alpha^{(3)}_{m,n} \;=\; & \frac{1}{2}\sum_{p,\lambda} C^{(p)}_{(m,\lambda)} C^{(3n-p-1)}_{(\lambda,m+1)} + \frac{1}{2}\sum_{p,\lambda} C^{(p)}_{(\lambda,m-\lambda)} C^{(3n-p-1)}_{(m-\lambda,\lambda+1)} \\
& + \frac{1}{2}\sum_{p,\lambda} C^{(p)}_{(\lambda,m)} C^{(3n-p-1)}_{(m+1,\lambda-1)} + \frac{1}{2}\sum_{p,\lambda} C^{(p)}_{(m+1,\lambda-1)} C^{(3n-p-1)}_{(\lambda,m)} \\
& + \frac{1}{2}\sum_{p,\lambda} C^{(p)}_{(\lambda,m+1)} C^{(3n-p-1)}_{(m,\lambda)} + \frac{1}{2}\sum_{p,\lambda} C^{(p)}_{(\lambda,m+1-\lambda)} C^{(3n-p-1)}_{(m-\lambda,\lambda)}
\end{aligned} \tag{3.57}$$

where the sums extend over $p = 0, 1, ..., 3n-1$ and $\lambda \geq 1$ such that the weights stay in Π_+, and each line corresponds to the terms coming from each line of (3.55). The summations in (3.57) can be rearranged into

$$\alpha^{(3)}_{m,n} \;=\; \sum_{p,\lambda} \left[C^{(p)}_{(m,\lambda)} C^{(3n-p-1)}_{(\lambda,m+1)} + C^{(p)}_{(\lambda,m-\lambda)} C^{(3n-p-1)}_{(m-\lambda,\lambda+1)} + C^{(p)}_{(\lambda,m)} C^{(3n-p-1)}_{(m+1,\lambda-1)} \right] \tag{3.58}$$

It is then a straightforward though tedious exercise in combinatorics to prove that

$$\alpha^{(3)}_{m,n} \;=\; C^{(3n)}_{(m,m)} - C^{(3n)}_{(m+2,m-1)} - C^{(3n)}_{(m-1,m+2)} + C^{(3n)}_{(m+1,m+1)} \tag{3.59}$$

for $m \geq 2$, by use of the definition (3.11) of the C's in terms of D's. For $m = 1$, we only have

$$\alpha^{(3)}_{1,n} \;=\; C^{(3)}_{(2,2)} \tag{3.60}$$

Finally, we write

$$\Delta^{(3)}_{3n}(\beta) \;=\; \prod_{m=1}^{n} (\mu_m)^{-\alpha^{(3)}_{m,n}} \tag{3.61}$$

and the result (3.31) follows from the definition of μ_m (2.26), with $a^{(3)}_{m,n} = \alpha^{(3)}_{m,n} - \alpha^{(3)}_{m+1,n}$, which amounts to (3.32).

4. SU(N) meander determinant

In this section, we present the generalization to $SU(N)$ of the notion of meander, through pairs of walk diagrams, in one to one correspondence with reduced elements of a particular ideal $\mathcal{I}_{Nn}^{(N)}(\beta)$ of the $SU(N)$ quotient $H_{Nn}^{(N)}(\beta)$ of the Hecke algebra $H_{Nn}(\beta)$, in which all antisymmetrizers of order $N+1$ vanish. The orthonormalization of a basis of this ideal yields a formula for the corresponding generalized meander determinant.

4.1. SU(N) walk diagrams

Let us denote by $\Lambda = \sum \lambda_i \omega_i = (\lambda_1, ..., \lambda_{N-1})$, $\lambda_i \in \mathbb{Z}$, the elements of the weight lattice of the $sl(N)$ algebra, generated by the fundamental weights ω_1, ω_2, ..., ω_{N-1} in $\mathbb{R}^{N-1}$, with the scalar products

$$\omega_i \cdot \omega_j \;=\; \frac{i(N-j)}{N} \tag{4.1}$$

for $1 \leq i \leq j \leq N-1$. The Weyl chamber $P_+ \subset P$ is defined as the set of weights

$$P_+ \;=\; \{(\lambda_1, ..., \lambda_{N-1}) \text{ such that } \lambda_i \geq 1 \text{ for all } i\} \tag{4.2}$$

The Weyl group of $sl(N)$ is the group generated by the reflections s_i w.r.t. the walls of the Weyl chamber, i.e., the hyperplanes $\lambda_i = 0$. It is isomorphic to the permutation group S_N of N objects. The Weyl chamber is nothing but the quotient of the weight lattice by the action of this group.

The weight lattice and Weyl chamber are made into simplices respectively denoted by Π and Π_+, by the adjunction of oriented links between the weights, along the vectors

$$\epsilon_i = \omega_i - \omega_{i-1}, \quad i = 2, 3, ..., N-1 \tag{4.3}$$

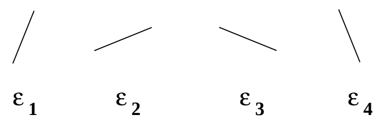
and $\epsilon_1 = \omega_1$, $\epsilon_N = -\omega_{N-1}$, with the property that $\sum \epsilon_i = 0$. Let us denote by $\rho = (1, 1, ..., 1)$ the origin (apex) of the Weyl chamber P_+.

A $SU(N)$ walk diagram of order Nn is a closed path of Nn steps on Π_+ starting and ending at ρ. It is uniquely determined by a sequence $\Lambda_0 = \rho$, Λ_1, ..., Λ_{Nn-1}, $\Lambda_{Nn} = \rho$ of weights in P_+, satisfying

$$\Lambda_i - \Lambda_{i-1} \in \{\epsilon_1, \epsilon_2, ..., \epsilon_N\} \tag{4.4}$$

for all $i = 1, 2, ..., Nn$. As before, the index i in Λ_i is referred to as the position of the weight Λ_i in the walk diagram. The set of $SU(N)$ walk diagrams of order Nn is denoted

by $W_{Nn}^{(N)}$. We can still represent the $SU(N)$ walk diagrams pictorially in the plane, by replacing each step ϵ_i by an edge of unit length, making an angle of $\frac{\pi}{2N}(N-2i+1)$ with the horizontal axis, and connecting the successive edges of each walk diagram. For illustration the $SU(4)$ edges read



(4.5)

We also define the fundamental $SU(N)$ walk diagram $a_0^{(N)}$ with the successive weights $\Lambda_{i+Nj} - \Lambda_{i+Nj-1} = \epsilon_i$, for $i = 1, 2, ..., N$ and $j = 0, 1, ..., n-1$.

To count the number $c_{Nn}^{(N)}$ of $SU(N)$ walk diagrams of order Nn, let us first compute the number $D_\Lambda^{(M)}$ of paths of M steps on Π from the origin $(0, 0, ..., 0)$ to a fixed weight Λ. Assume this path is made of n_1 steps ϵ_1, n_2 steps ϵ_2, ..., n_N steps ϵ_N, then we must have $\Lambda = \sum_{1 \leq i \leq N} n_i \epsilon_i$, and $n_1 + n_2 + ... + n_N = M$. This is easily inverted into

$$n_i \ = \ \epsilon_i \cdot \Lambda + \frac{M}{N} \tag{4.6}$$

for $i = 1, 2, ..., N$. The n_i are integers only if $M - \sum i\lambda_i = 0 \mod N$, otherwise there is no path of M steps from $(0, 0, ..., 0)$ to Λ, and $D_\Lambda^{(M)} = 0$. The paths are then obtained by arbitrarily choosing the n_i steps ϵ_i, resulting in

$$D_\Lambda^{(M)} \ = \ \frac{M!}{\prod_{i=1}^{N} (\frac{M}{N} + \epsilon_i \cdot \Lambda)!} \tag{4.7}$$

The number $c_{Nn}^{(N)}$ of $SU(N)$ walk diagrams of Nn steps is now obtained by subtracting from the number of closed paths on Π from the origin ρ to itself, the number of paths which cross the walls of the Weyl chamber, namely the hyperplanes $\lambda_i = 0$, $i = 1, 2, ..., N-1$. This is done by the following alternate sum over the images of the origin ρ of Π_+ under the action of the Weyl group of $sl(N)$ (isomorphic to S_N), generated by the reflections w.r.t. the walls of the Weyl chamber:

$$\begin{aligned} c_{Nn}^{(N)} \ &= \ \sum_{\sigma \in S_N} (-1)^{l(\sigma)} D_{\rho - \sigma(\rho)}^{(Nn)} \\ &= \ (Nn)! \prod_{i=0}^{N-1} \frac{i!}{(n+i)!} \end{aligned} \tag{4.8}$$

which gives a natural $SU(N)$ generalization of the Catalan numbers (2.1).

$N\backslash n$	1	2	3	4	5	6
2	1	2	5	14	42	132
3	1	5	42	462	6006	87516
4	1	14	462	24024	1662804	140229804
5	1	42	6006	1662804	701149020	396499770810
6	1	132	87516	140229804	396499770810	1671643033734960

Table V: The numbers $c_{Nn}^{(N)}$ of $SU(N)$ walk diagrams of order Nn, with $1 \leq n, N \leq 6$. The symmetry $N \leftrightarrow n$ will be interpreted later as some general duality property.

Similarly, the number $C_\Lambda^{(M)}$ of paths of M steps on Π_+ from the origin ρ to a given weight Λ is obtained by subtracting from the corresponding number of paths on Π, $D_{\rho,\Lambda}^{(M)} \equiv D_{\Lambda-\rho}^{(M)}$, the number of those which cross the walls of the Weyl chamber, namely

$$\begin{aligned} C_\Lambda^{(M)} &= \sum_{\sigma \in S_N} (-1)^{l(\sigma)} D_{\Lambda-\sigma(\rho)}^{(M)} \\ &= M! \frac{\prod_{1 \leq i < j \leq N} \big((\epsilon_i - \epsilon_j)\cdot\Lambda\big)!}{\prod_{i=0}^{N-1} \big(\frac{M}{N} + \epsilon_i \cdot (\Lambda - \rho) + i\big)!} \end{aligned} \tag{4.9}$$

where the second line follows from the celebrated Weyl character formula. Again, the number $C_\Lambda^{(M)}$ vanishes unless $M - \sum i(\lambda_i - 1) = 0 \text{ mod } N$.

4.2. Hecke algebra SU(N) quotient and ideal

We now concentrate on the quotient $H_{Nn}^{(N)}(\beta)$ of the Hecke algebra $H_{Nn}(\beta)$ (2.17), by the generalized Young antisymmetrizers of order $N+1$, namely defined by the conditions (2.17), supplemented by

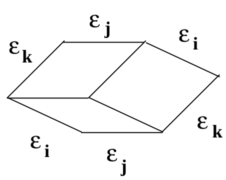
$$Y(e_i, e_{i+1}, ..., e_{i+N-1}) \ = \ 0 \tag{4.10}$$

for $i = 1, 2, ..., N(n-1)$. We now consider the left ideal $\mathcal{I}_{Nn}^{(N)}(\beta)$, generated by the element

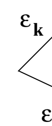
$$Y_{Nn}^{(N)} \ = \ \prod_{i=0}^{n-1} Y(e_{iN+1}, e_{iN+2}, ..., e_{iN+N-1}) \tag{4.11}$$

There is a one-to-one correspondence between the $SU(N)$ walk diagrams of order Nn and the reduced elements of $\mathcal{I}_{Nn}^{(N)}(\beta)$. To properly construct it, we first need to express

the $SU(N)$ walk diagrams as the results of successive box additions on the fundamental diagram $a_0^{(N)}$. Given a walk diagram $a \in W_{Nn}^{(N)}$, the process of box addition at position i on a, producing a diagram $b = a + \diamond_i$, is allowed only if a has a minimum at i, namely $N \geq r > s \geq 1$, if $\Lambda_{i+1} - \Lambda_i = \epsilon_s$ and $\Lambda_i - \Lambda_{i-1} = \epsilon_r$. The box addition amounts to replacing $\Lambda_i \to \Lambda_i + \epsilon_s - \epsilon_r$, i.e., exchanging the two steps ϵ_r and ϵ_s in the corresponding path on Π_+. In the above pictorial representation, a box addition amounts to adding to a a parallelogram (the "box"), with edges corresponding to the vectors ϵ_r and ϵ_s. This gives rise to $N(N-1)/2$ different types of boxes. It is clear that any walk $a \in W_{Nn}^{(N)}$ can be obtained from the fundamental one $a_0^{(N)}$ by successive box additions. As in the $SU(3)$ case, the box decomposition of a given walk a is not unique, due to all possible hexagon ambiguities. Indeed, for any three integers $N \geq i > j > k \geq 1$, there are two possibilities to change the succession of steps $(\epsilon_i, \epsilon_j, \epsilon_k)$ into $(\epsilon_k, \epsilon_j, \epsilon_i)$ by three successive box additions:

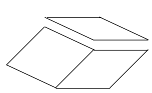


or



(4.12)

To resolve these ambiguities, we forbide all the box additions of the form



(4.13)

for all $N \geq i > j > k \geq 1$. With this last rule, each walk $a \in W_{Nn}^{(N)}$ has a unique box decomposition, represented as the set of boxes inbetween $a_0^{(N)}$ and a.

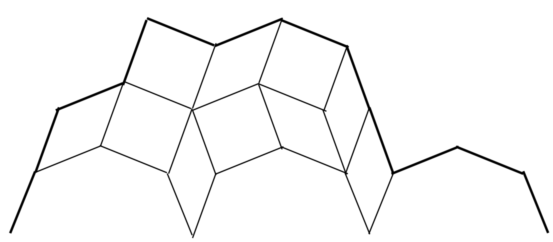


Fig. 4: A sample $SU(4)$ walk diagram a of order $4 \times 3 = 12$ is represented in thick line. It is made of a succession of steps of the form (4.5). We have indicated its box decomposition in thin lines, leading from the fundamental diagram $a_0^{(4)}$ of order 12 to a, after a total of $|a| = 10$ box additions.

For illustration, we display in Fig.4 a sample $SU(4)$ walk diagram of order 12, together with its box decomposition.

We can now construct the map φ from $W_{Nn}^{(N)}$ to the set of reduced elements of $\mathcal{I}_{Nn}^{(N)}(\beta)$, through

$$\varphi(a_0^{(N)}) \;=\; Y_{Nn}^{(N)} \tag{4.14}$$

with $Y_{Nn}^{(N)}$ as in (4.11), and the recursion on box additions, for any $a \in W_{Nn}^{(N)}$

$$\varphi(a+\diamond_i) \;=\; e_i\ \varphi(a) \tag{4.15}$$

This produces exactly once all the reduced elements of $\mathcal{I}_{Nn}^{(N)}(\beta)$. For illustration, the walk diagram of Fig.4, has the following image under φ: $e_4e_5e_6e_2e_3e_5e_6e_7e_4e_8Y(e_1,e_2,e_3)Y(e_5,e_6,e_7)Y(e_9,e_{10}$ as a result of 10 box additions on the fundamental diagram. As before, we introduce the basis 1, with elements

$$(a)_1 \;=\; (\alpha_N)^{-n/2}(\gamma_N)^n\ \varphi(a) \tag{4.16}$$

where α_N and γ_N are defined in (2.29). The normalization, somewhat arbitrary, is chosen for reasons which will become clear later.

The $SU(N)$ meanders are defined as pairs of walks $(a,b) \in W_{Nn}^{(N)}$, or equivalently of elements of this basis 1. To the latter, we attach the quantity $\big((a)_1,(b)_1\big)$, where the scalar product is attached to the Markov trace Tr on $H_{Nn}^{(N)}(\beta)$, defined by the normalization $\mathrm{Tr}(1) = (U_{N-1})^{Nn}$ and the recursion (2.7), with $\eta = \mu_{N-1} = U_{N-2}(\beta)/U_{N-1}(\beta)$. This leads to the $c_{Nn}^{(N)} \times c_{Nn}^{(N)}$ Gram matrix $\mathcal{G}_{Nn}^{(N)}(\beta)$, with entries

$$\big[\mathcal{G}_{Nn}^{(N)}(\beta)\big]_{a,b} \;=\; \big((a)_1,(b)_1\big) \tag{4.17}$$

for $a,b \in W_{Nn}^{(N)}$.

4.3. SU(N) determinant

The main result of this section is the following formula for the determinant $\Delta_{Nn}^{(N)}(\beta)$ of the Gram matrix (4.17):

$$\Delta_{Nn}^{(N)} \;=\; \prod_{m=1}^{n+N-2} \big[U_m(\beta)\big]^{a_{m,n}^{(N)}} \tag{4.18}$$

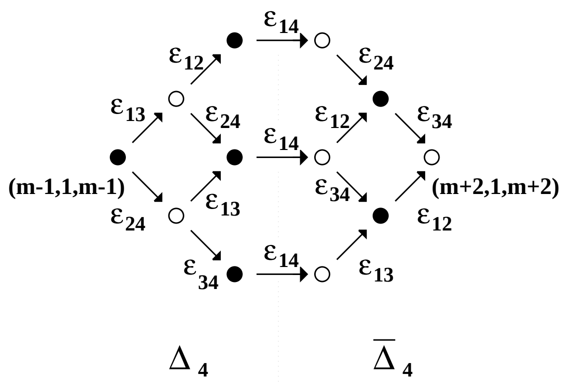


Fig. 5: The set of 12 vectors forming the difference operator defining $a_{m,n}^{(N)}$ in terms of $C_\Lambda^{(Nn)}$, for $N=4$. We have indicated the vectors $\epsilon_{i,j} \equiv \epsilon_i - \epsilon_j$ linking the dots, representing $(m-1,1,m-1)+u_k+v_l$ on the left half (Δ_4 operator), and $(m+2,1,m+2)-\bar{u}_k-\bar{v}_l$ on the right half ($\bar{\Delta}_4$ operator). The terms with a filled black circle come with a $+$, those with an empty circle with a $-$ in the final difference operator.

where $a_{m,n}^{(N)}$ are some integers, defined as follows.

We first introduce the vectors $u_0=v_0=\bar{u}_0=\bar{v}_0=0$, and

$$\begin{aligned} u_j &= j\epsilon_1 - (\epsilon_j + \epsilon_{j+1} + \ldots + \epsilon_{N-1}) \\ v_j &= -j\epsilon_N + (\epsilon_2 + \epsilon_3 + \ldots + \epsilon_{j+1}) \\ \bar{u}_j &= j(\epsilon_1 - \epsilon_N) - u_{N-1-j} \\ \bar{v}_j &= j(\epsilon_1 - \epsilon_N) - v_{N-1-j} \end{aligned} \tag{4.19}$$

for $j=1,2,\ldots,N-2$ (see Fig.5 for an illustration in the case $N=4$). We define the following difference operators Δ_N, and $\bar{\Delta}_N$ acting on any function $f(\alpha)$, of $\alpha \in P$ by the alternate sums

$$\begin{aligned} [\Delta_N f](\alpha) &= \sum_{\substack{i,j\geq 0 \\ i+j\leq N-2}} (-1)^{i+j}\ f(\alpha + (u_i + v_j)) \\ [\bar{\Delta}_N f](\alpha) &= \sum_{\substack{i,j\geq 0 \\ i+j\leq N-2}} (-1)^{i+j}\ f(\alpha - (\bar{u}_i + \bar{v}_j)) \end{aligned} \tag{4.20}$$

We also define Δ_N^* as the same expression as for Δ_N, except that the point $i=j=0$ is excluded from the sum (4.20). Now, we use the function $f(\Lambda)=C_\Lambda^{(Nn)}$, $\Lambda \in P_+$, and $C_\Lambda^{(Nn)}$ as in (4.9), to write the integers $a_{m,n}^{(N)}$ as

$$a_{m,n}^{(N)} = \Delta_N f\big(\rho + (m-N+2)(\epsilon_1 - \epsilon_N)\big) - \bar{\Delta}_N f\big(\rho + (m+1)(\epsilon_1 - \epsilon_N)\big) \tag{4.21}$$

for any integer $m \neq N-2$ and

$$a_{N-2,n}^{(N)} = \Delta_N^* f(\rho) - \bar{\Delta}_N f\big(\rho + (N-1)(\epsilon_1 - \epsilon_N)\big) \tag{4.22}$$

We have already noticed in the cases $N=2$ and 3 that the numbers $a_{m,n}^{(N)}$ take the form of an alternate sum over the Weyl group. Here we see that the correct generalization (4.21) is not an alternate sum over the Weyl group, but only over a set of $N(N-1)$ shifted weights, represented in Fig.5 for $N=4$. In this latter case, the corresponding 12 term-relation reads

$$\begin{aligned} a_{m,n}^{(4)} &= C_{(m-1,1,m-1)}^{(4n)} - C_{(m,2,m-2)}^{(4n)} - C_{(m-2,2,m)}^{(4n)} + C_{(m+2,1,m-2)}^{(4n)} \\ &+ C_{(m-2,1,m+2)}^{(4n)} + C_{(m-1,3,m-1)}^{(4n)} - C_{(m+3,1,m-1)}^{(4n)} - C_{(m-1,1,m+3)}^{(4n)} \\ &- C_{(m,3,m)}^{(4n)} + C_{(m+2,2,m)}^{(4n)} + C_{(m,2,m+2)}^{(4n)} - C_{(m+2,1,m+2)}^{(4n)} \end{aligned} \tag{4.23}$$

for $m \neq 2$, and

$$a_{2,n}^{(4)} = C_{(1,3,1)}^{(4n)} - C_{(5,1,1)}^{(4n)} - C_{(1,1,5)}^{(4n)} - C_{(2,3,2)}^{(4n)} + C_{(4,2,2)}^{(4n)} + C_{(2,2,4)}^{(4n)} - C_{(4,1,4)}^{(4n)} \tag{4.24}$$

The first few values for the integers $a_{m,n}^{(N)}$ are given in Tables VI-a,b for the cases $N=4,5$ respectively.

$m\backslash n$	1	2	3	4	5	6
1	1	20	627	24024	831402	-8498776
2	1	15	572	36036	2922504	274085526
3	1	22	880	48048	3375996	291900268
4		13	550	36036	2910876	265913626
5			341	24024	1951566	192203088
6				12012	1372104	139085738
7					492252	85314636
8						22064130

Table VI-a: The powers $a_{m,n}^{(4)}$ of U_m in the $SU(4)$ meander determinant $\Delta_{4n}^{(4)}(\beta)$, for $n=1,2,...,6$.

$m\backslash n$	1	2	3	4	5
1	1	69	10582	2494206	701149020
2	1	44	6435	2065908	1051723530
3	1	58	10712	3275220	1402298040
4	1	76	12311	3740340	1752872550
5		41	8736	3036846	1402298040
6			5278	1953504	1051723530
7				1170552	701149020
8					350574510

Table VI-b: The powers $a^{(5)}_{m,n}$ of U_m in the $SU(5)$ meander determinant $\Delta^{(5)}_{5n}(\beta)$, for $n = 1, 2, ..., 5$.

The formula (4.18) is proved by direct orthogonalization of the basis 1, namely the construction of a basis 2, with elements

$$(a)_2 \;=\; \sum_{\substack{b \in W^{(N)}_{Nn} \\ b \subset a}} P_{a,b} (b)_1 \tag{4.25}$$

where the inclusion $b \subset a$ means that a can be obtained from b by box additions, and such that $\big((a)_2, (b)_2\big) = \delta_{a,b}$. The basis 2 is constructed as follows. We start with

$$(a^{(N)}_0)_2 \;=\; (\alpha_N)^{n/2} (a)_1 \tag{4.26}$$

with α_N as in (2.29). This element has norm 1, due to (2.32), and the property

$$\mathrm{Tr}\big(Y(e_1, ..., e_{N-1})\big) \;=\; \gamma_N^{-1} \tag{4.27}$$

where the trace is taken over $H^{(N)}_N(\beta)$. This is readily proved by use of the recursion of the Y's (2.25). We finally get $\mathrm{Tr}\big((\gamma_N Y)^2\big) = 1$.

The other basis 2 elements are defined through the recursion relation, for any $a \in W^{(N)}_{Nn}$

$$(a + \diamond_{i,m})_2 \;=\; \sqrt{\frac{\mu_{m+1}}{\mu_m}} (e_i - \mu_m)\ (a)_2 \tag{4.28}$$

where i denotes as usual the position of the box addition, and m stands for the "height" of the box addition, defined as

$$m \;=\; \Lambda_i \cdot (\Lambda_{i+1} + \Lambda_{i-1} - 2\Lambda_i) \tag{4.29}$$

in terms of the weights Λ_{i-1}, Λ_i and Λ_{i+1} of respective positions $i-1$, i and $i+1$ on a. The basis 2 coincides with the restriction to $\mathcal{I}_{Nn}^{(N)}(\beta)$ of the semi-normal basis of [14] [16], and as such satisfies the condition

$$(a)_2^t(b)_2 \;=\; 0 \;\; \text{unless} \;\; a \;=\; b \tag{4.30}$$

Assuming (4.30), with exactly the same reasoning as in (3.43), it is easy to check that the normalization prefactor in (4.28) ensures that all the basis 2 elements have norm 1. This change of basis resolves the hexagon ambiguities, in the form of a straightforward generalization of (3.40), to all the possible hexagons (4.12).

As before, the $SU(N)$ meander determinant reads

$$\Delta_{Nn}^{(N)}(\beta) \;=\; \prod_{a \in W_{Nn}^{(N)}} P_{a,a}^{-2} \tag{4.31}$$

with P as in (4.25). In turn, we have

$$P^2_{a_0^{(N)},a_0^{(N)}} \;=\; (\prod_{i=1}^{N-1} (\mu_i)^{N-i})^n \tag{4.32}$$

as a direct consequence of (4.25) and (4.26) ($(a_0^{(N)})_2$ has norm 1), and the recursion relation

$$P^2_{a+\diamond_{i,m},a+\diamond_{i,m}} \;=\; \frac{\mu_{m+1}}{\mu_m} P_{a,a} \tag{4.33}$$

for all $a \in W_{Nn}^{(N)}$, solved as

$$P^2_{a,a} \;=\; (\mu_1^{N-1}\mu_2^{N-2} \cdots \mu_{N-1})^n \prod_{\substack{\text{all boxes} \\ \text{of } a}} \frac{\mu_{m+1}}{\mu_m} \tag{4.34}$$

As in the $SU(3)$ case, we rearrange the μ factors into edge weights. More precisely, to each edge (Λ, Λ') of a, corresponding to a step $\Lambda' - \Lambda = \epsilon_i$, we attach the weight

$$w(\Lambda, \Lambda') \;=\; \left(\prod_{l=i+1}^{N} \mu_{\Lambda \cdot (\epsilon_i - \epsilon_l)} \prod_{k=1}^{i-1} \mu_{\Lambda' \cdot (\epsilon_k - \epsilon_i)} \right)^{\frac{1}{2}} \tag{4.35}$$

in terms of which (4.34) is rewritten as

$$P^2_{a,a} \;=\; \prod_{\substack{\text{all steps}\\ v \text{ of } a}} w(v) \tag{4.36}$$

This is easily proved by induction on box additions, as the r.h.s. of (4.36) satisfies both (4.32) and (4.33). Indeed, the box addition $a \to a + \diamond_{i,m}$ transforms the sequence of weights of a $(\Lambda_{i-1}, \Lambda_i, \Lambda_{i+1})$, with say $\Lambda_i - \Lambda_{i-1} = \epsilon_s$ and $\Lambda_{i+1} - \Lambda_i = \epsilon_r$, into the sequence $(\Lambda_{i-1}, \Lambda'_i, \Lambda_{i+1})$ with $\Lambda'_i - \Lambda_{i-1} = \epsilon_r$; we then check that the edge weights satisfy $w(\Lambda_{i-1}, \Lambda'_i)w(\Lambda'_i, \Lambda_{i+1})/[w(\Lambda_{i-1}, \Lambda_i)w(\Lambda_i, \Lambda_{i+1})] = \mu_{m+1}/\mu_m$, with $m = \Lambda_i \cdot (\epsilon_r - \epsilon_s)$, hence (4.33) follows. After substitution of (4.36) into (4.31), we are left with an expression of the form

$$\Delta^{(N)}_{Nn}(\beta) \;=\; \prod_{m=1}^{N+n-1} \mu_m^{-\alpha^{(N)}_{m,n}} \tag{4.37}$$

To compute $\alpha^{(N)}_{m,n}$, we have to enumerate the walks containing edges, whose weight contains a μ_m. According to the definition (4.35), such an edge is of the form (Λ, Λ'), $\Lambda' - \Lambda = \epsilon_i$, with either $\Lambda' \cdot (\epsilon_k - \epsilon_i) = m$ for some $k < i$ or $\Lambda \cdot (\epsilon_i - \epsilon_l) = m$ for some $l > i$. To proceed, we must count the number of edges of walks of $W^{(N)}_{Nn}$ with specified ends (Λ, Λ'), at positions, say p and $p+1$. This edge cuts the walk a into two portions

(i) a path of p steps from the origin ρ to Λ on Π_+. There are $C^{(p)}_\Lambda$ such paths (see (4.9)).

(ii) a path of $Nn - p - 1$ steps from Λ' to ρ on Π_+. Upon reversal of all the orientations of its links (namely exchanging $\epsilon_i \leftrightarrow \epsilon_{N+1-i}$), it can be viewed as the reversed path of $Nn - p - 1$ steps from ρ to Λ'^t (where $(\lambda_1, \lambda_2, ..., \lambda_{N-1})^t = (\lambda_{N-1}, \lambda_{N-2}, ..., \lambda_1)$). There are $C^{(Nn-p-1)}_{\Lambda'^t}$ such paths (see (4.9)).

We may now compute the numbers $\alpha^{(N)}_{m,n}$ of (4.37), with the result

$$\alpha^{(N)}_{m,n} \;=\; \sum_{p=0}^{Nn-1} \sum_{1 \le i < l \le N} \sum_{\substack{\Lambda \in P_+\\ \Lambda \cdot (\epsilon_i - \epsilon_l) = m}} C^{(p)}_\Lambda C^{(Nn-p-1)}_{(\Lambda+\epsilon_i)^t} \tag{4.38}$$

where we have assembled all the contributions from the weight factors $w(\Lambda, \Lambda + \epsilon_i)$ as well as those from $w(\Lambda' - \epsilon_{N+1-i}, \Lambda')$, by noting that $(\Lambda' + \epsilon_i)^t = \Lambda' - \epsilon_{N+1-i}$.

The final formula (4.18) follows from the definition (2.26) of μ_m and $a^{(N)}_{m,n} = \alpha^{(N)}_{m,n} - \alpha^{(N)}_{m+1,n}$. Let us first compute the numbers $\alpha^{(N)}_{m,n}$ (4.38). We can use the expression (4.9) for the C's in terms of the D's, which are multinomial coefficients, to evaluate the various

sums in (4.38). This finally leads to the following result. We will use the definition (4.20) for the difference operator Δ_N. Let us also define the vectors $w_0 = 0$ and

$$w_j \;=\; \epsilon_2 + 2\epsilon_3 + ... + (j-1)\epsilon_{j-1} + j(\epsilon_j + \epsilon_{j+1} + ... + \epsilon_N) \tag{4.39}$$

for $j = 1, 2, ..., N$. Then, using the function $f(\Lambda) = C_\Lambda^{(Nn)}$, the integer $\alpha_{m,n}^{(N)}$ reads

$$\alpha_{m,n}^{(N)} \;=\; \sum_{\substack{p\geq 0\\ 2p\leq N-2}} \Delta_{N-2p} f(\rho + (m-N+2)(\epsilon_1 - \epsilon_N) + w_p) \tag{4.40}$$

for $m \geq N-1$. When $1 \leq m \leq N-2$, we simply have to omit the term $i = j = 0$ in Δ_N. After substitution of (4.20), the alternate sum on the r.h.s. of (4.40) extends over $N(N^2-1)/6$ terms of the form $C_\Lambda^{(Nn)}$, $\Lambda = \rho + (m-N+2)(\epsilon_1 - \epsilon_N) + u_i + v_j + w_p$, $0 \leq i, j \leq N-2$, $0 \leq 2p \leq N-2$ hence forming a "pyramid" of weights. The result (4.21) for the numbers $a_{m,n}^{(N)} = \alpha_{m,n}^{(N)} - \alpha_{m+1,n}^{(N)}$ then follows from many cancellations between the pyramids of terms of (4.40) pertaining to m and $m+1$, leaving us eventually with only $N(N-1)$ terms (see Fig.5). Eq.(4.22) corresponds to the omission of the term $i = j = 0$ in (4.40) for $m = 1, ..., N-2$.

4.4. Duality

In this section, we describe a duality relation between the $SU(N)$ and the $SU(k)$ meanders of same order Nk. This results in a duality formula for the corresponding meander determinants.

The compact definitions (4.26) and (4.28) for the basis 2 elements of $\mathcal{I}_{Nn}^{(N)}(\beta)$ lead us to a simple formula, relating the $SU(N)$ meander determinant of order Nk to the $SU(k)$ meander determinant of same order, namely

$$\Delta_{Nk}^{(N)}(\beta)\Delta_{kN}^{(k)}(\beta) \;=\; (\Phi_{N,k})^{c_{Nk}^{(N)}} \tag{4.41}$$

where, $\Phi_{N,k}$ is symmetric in $k \leftrightarrow N$, and for $k \leq N$

$$\Phi_{N,k} \;=\; \prod_{m=1}^{k-1} (U_m)^{m+1} \prod_{m=k}^{N-1} (U_m)^k \prod_{m=N}^{N+k-2} (U_m)^{N+k-1-m} \tag{4.42}$$

in terms of the Chebishev polynomials (2.14). Note also that from the definition (4.8),

$$c_{Nk}^{(N)} \;=\; (Nk)! \frac{\prod_{i=1}^{k-1} i! \prod_{i=1}^{N-1} i!}{\prod_{i=1}^{N+k-1} i!} \;=\; c_{Nk}^{(k)} \tag{4.43}$$

which makes the r.h.s. of (4.41) symmetric under $k \leftrightarrow N$.

The duality formula (4.41) gives a number of combinatorial identities relating the numbers $a_{m,k}^{(N)}$ and $a_{m,N}^{(k)}$ (4.21)(4.22), namely that, for $k \leq N$

$$a_{m,k}^{(N)} + a_{m,N}^{(k)} = \begin{cases} (m+1)c_{Nk}^{(N)} & (\text{if } m < k) \\ kc_{Nk}^{(N)} & (\text{if } k \leq m < N) \\ (N+k-m-1)c_{Nk}^{(N)} & (\text{if } N \leq m < N+k-1) \end{cases} \tag{4.44}$$

including the cases (4.22) when $m = N-2$ or $k-2$.

As an example, let us take $N = 3$ and $k = 2$, in which cases we have (see Tables II and IV)

$$\begin{aligned} \Delta_6^{(3)} &= U_1^6 U_2^6 U_3^4 \\ \Delta_6^{(2)} &= U_1^4 U_2^4 U_3 \end{aligned} \tag{4.45}$$

respectively from (3.35) and (2.13), with $C_3^{(6)} = \binom{6}{2} - \binom{6}{1} = 9$, $C_5^{(6)} = \binom{6}{1} - 1 = 5$, $C_7^{(6)} = 1$. We check that

$$\Delta_6^{(3)} \Delta_6^{(2)} = (U_1^2 U_2^2 U_3)^5 \tag{4.46}$$

which amounts to (4.41)(4.42), with $c_6^{(3)} = c_6^{(2)} = 5$ and $\Phi_{3,2} = U_1^2 U_2^2 U_3$. More generally, we can check the above duality relation on the various Tables II, IV and VI-a,b, by using also the Table V for the numbers $c_{Nn}^{(N)}$.

A simple consequence of (4.41) is that the "self-dual" determinants, with $k = N$, read

$$\begin{aligned} \Delta_{N^2}^{(N)}(\beta) &= (\Phi_{N,N})^{c_{N^2}^{(N)}/2} \\ &= (U_1^2 U_2^3 ... U_{N-2}^{N-1} U_{N-1}^N U_N^{N-1} ... U_{2N-3}^2 U_{2N-2})^{c_{N^2}^{(N)}/2} \end{aligned} \tag{4.47}$$

which is considerably simpler than (4.18) (4.21). This is readily checked for $N = 2, 3, 4, 5$ on Tables II, IV and VI-a,b.

The duality formula (4.41) is a consequence of the existence of a duality map δ between the basis 2 elements of $\mathcal{I}_{Nk}^{(N)}(\beta)$ and $\mathcal{I}_{Nk}^{(k)}(\beta)$, or equivalently between their labels $W_{Nk}^{(N)}$ and $W_{Nk}^{(k)}$. The map δ is defined as follows. First we need to define the *maximal* walk diagram $a_{max}^{(k)} \in W_{Nk}^{(k)}$, as the walk with N steps ϵ_1, followed by N steps ϵ_2, ..., followed by N steps ϵ_k. In other words, the weights of this walk are

$$\Lambda_{Ni+j} = N(\epsilon_1 + \epsilon_2 + ... + \epsilon_i) + j\epsilon_{i+1} \tag{4.48}$$

for $i = 0, 1, ..., k-1$ and $j = 1, 2, ..., N$. This walk is maximal w.r.t. box additions, as it has no minimum, hence no extra box can be added to it. We also need to define the

concept of box subtraction for elements of the basis 2: we will say that $(b)_2$ is the result of a box subtraction at position i and height m on $(a)_2$, and write that $(b)_2 = (a - \diamond_{i,m})_2$, if $(a)_2 = (b + \diamond_{i,m})_2$ is the result of a box addition at position i and height m on b (c.f. (4.28)). We will use the same terminology for the corresponding walk diagrams.

The duality map $\delta \ : \ W_{Nk}^{(N)} \to W_{Nk}^{(k)}$ is defined by

$$\delta(a_0^{(N)}) \ = \ a_{max}^{(k)} \tag{4.49}$$

and the recursion relation

$$\delta(a + \diamond_{i,m}) \ = \ \delta(a) - \diamond_{i,m} \tag{4.50}$$

In other words, the recursion adds successive boxes on $a_0^{(N)}$ which it subtracts accordingly from $a_{max}^{(k)}$. To prove that δ is well defined, we must simply check that each minimum on a is a "maximum" on $\delta(a)$, i.e., a position at which a box can be subtracted. This is clear on $a_0^{(N)}$ and its image, as the $k-1$ minima of $a_0^{(N)}$ lie at positions $i = N, 2N, ..., (k-1)N$, equal to the positions of the maxima on $a_{max}^{(k)}$, namely the transitions between the steps $\epsilon_i \to \epsilon_{i+1}$, $i = 1, 2, ..., k-1$. The recursion then makes it clear that whenever a minimum is created on a by a neighbouring box addition, the corresponding box subtraction on b creates a maximum. Moreover, as $|W_{Nk}^{(N)}| = c_{Nk}^{(N)} = c_{Nk}^{(k)} = |W_{Nk}^{(k)}|$ (4.43), δ is a bijection.

The computation of the determinants $\Delta_{Nk}^{(N)}(\beta)$ and $\Delta_{Nk}^{(k)}(\beta)$ involves a product over the quantities $P_{a,a}^{-2}$ defined by (4.32) (4.33). By a slight abuse of notation, we will denote indifferently by $P_{a,a}$ the matrix elements for both $SU(N)$ and $SU(k)$ cases, simply distinguished by the fact that $a \in W_{Nk}^{(N)}$ or $W_{NK}^{(k)}$. Let us prove that, for all $a \in W_{Nk}^{(N)}$

$$P_{a,a}^2 P_{\delta(a),\delta(a)}^2 \ = \ P^2_{a_0^{(N)},a_0^{(N)}} P^2_{a_{max}^{(k)},a_{max}^{(k)}} \tag{4.51}$$

This is readily done by induction on box additions on a, as

$$\begin{aligned} P^2_{a+\diamond_{i,m},a+\diamond_{i,m}} P^2_{\delta(a+\diamond_{i,m}),\delta(a+\diamond_{i,m})} \ &= \ P^2_{a+\diamond_{i,m},a+\diamond_{i,m}} P^2_{\delta(a)-\diamond_{i,m},\delta(a)-\diamond_{i,m}} \\ &= \ \frac{\mu_{m+1}}{\mu_m} P^2_{a,a} \frac{\mu_m}{\mu_{m+1}} P^2_{\delta(a),\delta(a)} \\ &= \ P^2_{a,a} P^2_{\delta(a),\delta(a)} \end{aligned} \tag{4.52}$$

where we have successively used the recursive definition (4.50) of δ and the recursion (4.33) for both $P^2_{a+\diamond,a+\diamond}$ and $P^2_{b+\diamond,b+\diamond}$, with $b = \delta(a) - \diamond$. Eq.(4.51) follows.

Therefore, the product of meander determinants reads

$$\begin{aligned}\Delta_{Nk}^{(N)}(\beta)\Delta_{Nk}^{(k)}(\beta) &= \prod_{a\in W_{Nk}^{(N)}} P_{a,a}^{-2} \prod_{b\in W_{Nk}^{(k)}} P_{b,b}^{-2} \\ &= \prod_{a\in W_{Nk}^{(N)}} P_{a,a}^{-2} P_{\delta(a),\delta(a)}^{-2} \\ &= \left(P^{-2}_{a_0^{(N)},a_0^{(N)}} P^{-2}_{a_{max}^{(k)},a_{max}^{(k)}}\right)^{c_{Nk}^{(N)}}\end{aligned} \tag{4.53}$$

as $|W_{Nk}^{(N)}| = c_{Nk}^{(N)}$. The formula (4.41) follows from (4.53), with $\Phi_{N,k}^{-1} = P^2_{a_0^{(N)},a_0^{(N)}} P^2_{a_{max}^{(k)},a_{max}^{(k)}}$. The first factor (4.32) is known. The second reads, from (4.36)

$$P^2_{a_{max}^{(k)},a_{max}^{(k)}} = \prod_{\substack{\text{all steps} \\ v \text{ of } a_{max}^{(k)}}} w(v) \tag{4.54}$$

with the weights w as in (4.35) for $N \to k$ and $n \to N$, and the steps as in (4.48). Assembling all the powers of μ, we find

$$P^2_{a_{max}^{(k)},a_{max}^{(k)}} = \prod_{j=1}^{N} \prod_{i=j}^{j+k-2} (\mu_i)^{k+j+1-i} \tag{4.55}$$

hence finally, for $k \leq N$

$$\begin{aligned}\Phi_{N,k} &= \prod_{m=1}^{N-1} (\mu_m)^{-k(N-m)} \prod_{j=1}^{N} \prod_{i=j}^{j+k-2} (\mu_i)^{i-(k+j+1)} \\ &= \prod_{m=1}^{k-1} (\mu_m)^{\frac{m(m+1)}{2} - Nk} \prod_{m=k}^{N} (\mu_m)^{k(m-N-\frac{k-1}{2})} \prod_{m=N+1}^{N+m-2} (\mu_m)^{\frac{(m-k-N)(k+N-m-1)}{2}}\end{aligned} \tag{4.56}$$

Using the definition (2.26), this is easily translated into the final result (4.42).

4.5. Duality and Young tableaux

This duality is yet another manifestation of the level-rank duality of the affine Lie algebras $\widehat{sl(n)}_k \leftrightarrow \widehat{sl(k)}_n$ [17], through which integrable representations, attached to Young tableaux of at most n rows and k columns ($\widehat{sl(n)}_k$) are mapped onto the dual (transposed) ones with at most k rows and n columns ($\widehat{sl(k)}_n$). A direct way to understand this duality, is provided by the standard formulation [14] [16] of the basis 2, namely by the use of a mapping between the basis 2 elements and the standard Young tableaux which have the

shape of a rectangle of N rows by k columns (basis of $\mathcal{I}_{Nk}^{(N)}(\beta)$) sent by transposition to the standard Young tableaux having the shape of a rectangle of k rows by N columns (basis of $\mathcal{I}_{Nk}^{(k)}(\beta)$).

A standard Young tableau of given shape S, S a Young tableau of M boxes (i.e. an arrangement of say r rows of repectively l_1, l_2,...,l_r square boxes, with $l_1 \geq l_2 \geq ... \geq l_r \geq 1$ and $l_1 + l_2 + ... + l_r = M$), consists of the tableau S, together with a labelling (marking) of the boxes of S, using exactly once each of the integers $1, 2, ..., M$, and such that the labels are strictly increasing along the rows (from left to right) and along the columns (from top to bottom).

In the particular case of a rectangular shape S with $r = N$, $l_1 = l_2 = ... = l_N = k$, the set $S_{N,k}$ of the corresponding standard tableaux is in bijection with the set of $SU(N)$ walk diagrams of order Nk. Indeed let us define the map $f \; : \; S_{N,k} \to W_{Nk}^{(N)}$, by sending any standard tableau with N rows and k columns to the walk with successive steps v_i, $i = 1, 2, ..., Nk$ defined by

$$v_i \;=\; \epsilon_{r(i)} \tag{4.57}$$

where $r(i)$ denotes the number of the row of the box marked i in the standard tableau. For instance, the tableau whose marks are entered by successive columns $(1, 2, ..., N)$, $(N+1, N+2, ..., 2N)$, ... $((k-1)N, (k-1)N+1, ..., kN)$ is sent to the fundamental walk $a_0^{(N)}$, whereas the tableau whose marks are entered by successive lines $(1, 2, ..., k)$, $(k+1, k+2, ..., 2k)$, ..., $((N-1)k+1, (N-1)k+2, ..., Nk)$ is sent to the maximal one $a_{max}^{(N)}$.

The map f is clearly invertible, as we may fill the rectangular shape as we move along any walk a, the i-th mark corresponding to the i-th step, say $v_i = \epsilon_j$, and being made in the leftmost available (unmarked) box of the j-th row, thus filling eventually the whole tableau, as there is an equal total number k of steps of each kind ϵ_1, ..., ϵ_N.

The process of box addition at position i on a is interpreted in the standard tableau $f^{-1}(a)$ as the interchange of the markings i and $i+1$ if $i+1$ is in a strictly earlier row than i (with $r(i+1) < r(i)$), this being only possible if the ordering of rows and columns is preserved by the interchange: this corresponds exactly to the situation where the original walk has minimum at position i.

Now we see that the duality map δ has the simple interpretation as transposition, namely interchange of rows and columns, in the standard tableau picture, namely $f^{-1}(\delta(a)) = f^{-1}(a)^t$, for all $a \in W_{NK}^{(N)}$. Hence the map $f^{-1} \circ \delta \circ f$ is nothing but the

tableau transposition, which maps $S_{N,k} \to S_{k,N}$. The dual correspondence (4.50) between box additions and subtractions becomes clear with the above interpretation: the interchange between the marks i and $i+1$ has the effect of a box addition on a standard tableau iff it has the effect of a box subtraction on the transposed tableau.

5. Hecke determinants

In this section, we present determinant formulae for the natural generalization of meander determinants to the whole $SU(N)$ quotient $H_n^{(N)}(\beta)$ of the Hecke algebra. This provides yet another direction of generalization of meanders.

5.1. Bases of the Hecke algebra

The standard basis [14] [15] [16] of the Hecke algebra $H_n(\beta)$ is indexed by pairs (s_1, s_2) of standard tableaux of n boxes with the same shape S, where S describes the set of Young tableaux of n boxes. As already mentioned, this basis uses the description of the Hecke algebra (2.17) in terms of the generators T_i (2.18). The restriction of this basis to the quotient $H_n^{(N)}(\beta)$ is simply obtained by restricting the shapes S to the tableaux with at most N rows.

Let us present now a slightly different basis of $H_n^{(N)}(\beta)$, which we call basis 1 by analogy with the previous sections. This basis 1 will be indexed by pairs of *open* walk diagrams, rather than standard tableaux; the two objects are however in one-to-one correspondence.

For any given weight $\Lambda \in P_+$, an *open* walk diagram of order n ending at Λ is a path of n steps on Π_+, starting at the origin ρ and ending at Λ. In particular, we must have $n - \sum i(\lambda_i - 1) = 0 \bmod N$, if $\Lambda = (\lambda_1, ..., \lambda_{N-1})$. Let us denote by W_Λ^n the set of open $SU(N)$ walk diagrams of order n, ending at Λ. Writing

$$\begin{aligned} \Lambda &= \rho + \sum_{i=1}^{N} l_i \epsilon_i \\ n &= \sum_{i=1}^{N} l_i \end{aligned} \tag{5.1}$$

easily inverted into $l_i = (\Lambda - \rho) \cdot \epsilon_i + n/N$ (see (4.6)), we may identify each walk diagram $a \in W_\Lambda^n$ with a standard tableau whose shape is the Young tableau with l_i boxes in the i-th row, $1 \leq i \leq N$. Indeed, the marking of the boxes corresponding to a is performed as one moves along the path; say when the i-th step is made, with $v_i = \epsilon_j$, we mark with

the integer i the leftmost available (non-marked) box in the j-th row. We have already computed in (4.9) the number $C_\Lambda^{(n)}$ of open walk diagrams of order n ending at Λ.

The open walks of W_Λ^n can be generated by box additions on the fundamental one, denoted $a_0^{(n,\Lambda)}$, with steps

$$\begin{aligned}
v_{Ni+j} &= \epsilon_j \quad \text{for} \begin{cases} i = 0,1,...,l_N-1 \\ j = 1,2,...,N \end{cases} \\
v_{Nl_N+(N-1)i+j} &= \epsilon_j \quad \text{for} \begin{cases} i = 0,1,...,l_{N-1}-l_N-1 \\ j = 1,2,...,N-1 \end{cases} \\
\cdots \quad & \quad \cdots \\
v_{Nl_N+(N-1)(l_{N-1}-l_N)+...+2(l_2-l_3)+i+1} &= \epsilon_1 \quad \text{for } i = 0,1,...,l_1-l_2-1
\end{aligned} \tag{5.2}$$

(with l_i defined by (5.1)), which corresponds to entering the successive marks of the associated Young tableau by columns. A box addition at position $i \in \{1,2,...,n-1\}$ on $a \in W_\Lambda^n$, denoted by $a \to a + \diamond_i$, is defined in the same way as before (see Sect.4.2), and we still resolve the hexagon ambiguities by forbiding the box additions of the form (4.13). This permits to construct all the walks of W_Λ^n by successive box additions on the fundamental $a_0^{(n,\Lambda)}$.

We are now ready to define the basis 1 elements of $H_n^{(N)}(\beta)$. They are labelled by pairs (a,b) of open walk diagrams belonging to the same set W_Λ^n, where Λ runs over P_+. We start with the fundamental element

$$\begin{aligned}
(a_0^{(n,\Lambda)}, a_0^{(n,\Lambda)})_1 &= \prod_{i=0}^{l_N-1} E(e_{Ni+1}, e_{Ni+2}, ...e_{Ni+N-1}) \\
&\times \prod_{i=0}^{l_{N-1}-l_N-1} E(e_{Nl_N+(N-1)i+1}, ..., e_{Nl_N+(N-1)i+N-2}) \\
&\times \quad ... \\
&\times \prod_{i=0}^{l_2-l_3-1} E(e_{Nl_N+(N-1)(l_{N-1}-l_N)+...+3(l_3-l_4)+2i+1})
\end{aligned} \tag{5.3}$$

where the antisymmetrizer E is defined in (2.24), and related to Y through (2.28)(2.29). This product form corresponds to the column-preserving antisymmetrizer of [14] [15].

The other basis 1 elements are defined recursively using box additions on either walk diagram of the pair $(a,b) \in W_\Lambda^n$, namely

$$\begin{aligned}
(a + \diamond_i, b)_1 &= e_i (a,b)_1 \\
(a, b + \diamond_j)_1 &= (a,b)_1 e_j
\end{aligned} \tag{5.4}$$

We will call left (resp. right) box additions those pertaining to the first (resp. second) line of (5.4). Note that the forbidden additions (4.13) make the box decompositions of both a and b unique, and so is $(a,b)_1$.

For illustration, let us describe the basis 1 for $H_3^{(3)}$. There are three types of open walk diagrams of 3 steps, namely those which end at the $SU(3)$ weights $(1,1)$, $(2,2)$ or $(4,1)$, with $|W^3_{(1,1)}| = 1$, $|W^3_{(2,2)}| = 2$ and $|W^3_{(4,1)}| = 1$. With $E(e_1,e_2) = Y(e_1,e_2)$ and $E(e_1) = e_1$, the basis 1 elements read respectively

(5.5)

Note that the basis 1 for $H_3^{(2)}$ is simply obtained by imposing the vanishing of the antisymmetrizer of order 3, namely by erasing the first line in (5.5): it consists of the five elements e_1, e_2e_1, e_1e_2, $e_2e_1e_2 = e_2$ and 1.

As an immediate consequence, we get a formula for the dimension of $H_n^{(N)}(\beta)$ as a vector space, namely

$$\dim\left(H_n^{(N)}(\beta)\right) \;=\; \sum_{\Lambda\in P_+} \left(C_\Lambda^{(n)}\right)^2 \tag{5.6}$$

by enumerating all the pairs of open $SU(N)$ walk diagrams of order n. The first few of these dimensions are displayed in Table VII.

$N\backslash n$	1	2	3	4	5	6	7	8	9	10
2	1	2	5	14	42	132	429	1430	4862	16796
3	1	2	6	23	103	513	2761	15767	94359	586590
4	1	2	6	24	119	694	4582	33324	261808	2190688
5	1	2	6	24	120	719	5003	39429	344837	3291590
6	1	2	6	24	120	720	5039	40270	361302	3587916

Table VII: The dimensions $\dim\left(H_n^{(N)}(\beta)\right)$ of the $SU(N)$ quotients of the Hecke algebra $H_n(\beta)$, with $2 \leq N \leq 6$ and $1 \leq n \leq 10$.

The Gram matrix $\mathcal{H}_n^{(N)}(\beta)$ of the basis 1 of $H_n^{(N)}(\beta)$ reads

$$\left[\mathcal{H}_n^{(N)}(\beta)\right]_{(a,b),(c,d)} \;=\; \left((a,b)_1,(c,d)_1\right) \tag{5.7}$$

for (a,b) and $(c,d) \in \cup_{\Lambda\in P_+}(W_\Lambda^n)^2$. We wish to compute the determinant of this matrix exactly. In the case $n=3=N$ of (5.5), this Gram matrix reads

$$\mathcal{H}_3^{(3)}(\beta) \;=\; (\beta^2-1)\begin{pmatrix} \beta^2(\beta^2-1) & \beta^2 & \beta^3 & \beta^3 & \beta^4 & \beta \\ \beta^2 & \beta^2(\beta^2-1) & \beta^3 & \beta^3 & 2\beta^2 & \beta(\beta^2-1) \\ \beta^3 & \beta^3 & \beta^4 & 2\beta^2 & 2\beta^3 & \beta^2 \\ \beta^3 & \beta^3 & 2\beta^2 & \beta^4 & 2\beta^3 & \beta^2 \\ \beta^4 & 2\beta^2 & 2\beta^3 & 2\beta^3 & 2\beta^4 & \beta^3 \\ \beta & \beta(\beta^2-1) & \beta^2 & \beta^2 & \beta^3 & (\beta^2-1)^2 \end{pmatrix} \tag{5.8}$$

The semi-normal basis of the Hecke algebra presented in [14] and [16] restricts up to normalization factors to the orthonormal basis 2 w.r.t. the scalar product (,). In our language, the basis 2 is constructed as follows. For each $\Lambda \in P_+$, we introduce the fundamental element

$$(a_0^{(n,\Lambda)}, a_0^{(n,\Lambda)})_2 \;=\; \left(\prod_{\substack{\text{all steps} \\ v \text{ of } a_0^{(n,\Lambda)}}} w(v)\right) g_\Lambda\; E_\Lambda(e_1,e_2,\ldots,e_{n-1}) \tag{5.9}$$

where, g_Λ is a normalization constant and E_Λ are the orthogonal idempotents of the semi-normal basis [14] [16], defined as follows, in terms of the Murphy operators L_m (2.22). We have

$$E_\Lambda \;=\; \prod_{m=2}^{n} \prod_{\substack{-m<p<m \\ p\neq 0 \text{ if } m=2,3}} \frac{L_p-[p]_q}{[r_\Lambda(m)]_q-[p]_q} \tag{5.10}$$

where $[p]_q = 1 + q + q^2 + ... + q^{p-1}$ and $[-p]_q = -(q^{-1} + q^{-2} + ... + q^{-p})$ for $p > 0$, and $[0]_q = 0$, and $r_\Lambda(m) = j - i$ if

$$l_1 + l_2 + ... + l_{i-1} < m \leq l_1 + l_2 + ... + l_i \tag{5.11}$$

for l_i as in (5.1), and $j = m - (l_1 + ... + l_{i-1})$. In the standard tableau formulation of Λ (with marks entered by successive columns), i and j are respectively the numbers of the row and column in which the mark m occurs. Moreover, in (5.9), the normalization constant g_Λ will eventually ensure that the basis 2 element has norm 1. Let us comment briefly on this normalization now.

Note first the existence of the "inclusion" order on W_Λ^n, which we now denote by $a \leq b$ iff $a \subset b$, namely if b can be obtained from a by some box additions. Moreover, this order is extended to all open walk diagrams by deciding that $W_\Lambda^n \leq W_{\Lambda'}^n$ (we also write $\Lambda \leq \Lambda'$) iff the weight Λ' can be obtained from Λ by successive "pushes" $p_{i,j}$, $N \geq i > j \geq 1$, defined as

$$p_{i,j}(\Lambda) \;=\; \Lambda + \epsilon_i - \epsilon_j \tag{5.12}$$

allowed only if the result is still in P_+. In the Young tableau formulation of Λ, this amounts to "pushing" the rightmost box in the j-th row to the i-th row, and is allowed only if the result is still a Young tableau. This gives an order $\leq$ on all open walk diagrams. The change of basis $1 \to 2$ will be triangular with respect to $\leq$, namely

$$(a,b)_2 \;=\; \sum_{c \leq a \;,\; d \leq b} P_{(a,b),(c,d)}(c,d)_1 \tag{5.13}$$

The normalization g_Λ is chosen so that

$$g_\Lambda E_\Lambda \;=\; (a_0^{(n,\Lambda)}, a_0^{(n,\Lambda)})_1 + \sum_{\substack{\Lambda' \leq \Lambda \\ \Lambda' \neq \Lambda}} \sum_{a,b \in W_{\Lambda'}^n} \nu_{\Lambda,\Lambda'}^{(a,b)} \; (a,b)_1 \tag{5.14}$$

for some coefficients $\nu_{\Lambda,\Lambda'}^{(a,b)}$. The condition (5.14) will enable us to orthogonalize the Gram matrix $\mathcal{H}_n^{(N)}(\beta)$ by the Gram-Schmidt procedure, through a triangular redefinition of its lines and columns. The value of g_Λ can be found for instance in [16] and reads

$$g_\Lambda \;=\; (\alpha_N)^{-l_N} (\alpha_{N-1})^{l_N - l_{N-1}} ... (\alpha_2)^{l_3 - l_2} \tag{5.15}$$

in terms of the numbers α_j (2.29) and the integers l_j (5.1).

The other basis 2 elements are obtained by (left and right) box additions on the fundamental elements (5.9), according to the following recursions

$$
\begin{aligned}
(a+\diamond_{i,m},b)_2 &= \sqrt{\frac{\mu_{m+1}}{\mu_m}}(e_i-\mu_m)(a,b)_2 \\
(a,b+\diamond_{j,l})_2 &= \sqrt{\frac{\mu_{l+1}}{\mu_l}}(a,b)_2(e_j-\mu_l)
\end{aligned}
\tag{5.16}
$$

where m and l denote the heights of the box additions (4.29). The normalization of the basis 2 elements to unity is a consequence of their orthogonality, in the same way as before (see (3.43)).

In the case of $H_3^{(3)}$, the normalized idempotents $g_\Lambda E_\Lambda$ read, in terms of the e_i

$$
\begin{aligned}
g_{(1,1)}E_{(1,1)} &= Y(e_1,e_2) \\
g_{(2,2)}E_{(2,2)} &= e_1-\mu_1\mu_2 Y(e_1,e_2) \;=\; \mu_2 e_1(1-\mu_1 e_2)e_1 \\
g_{(4,1)}E_{(4,1)} &= (1-\mu_1 e_1)(1-\mu_2 e_2)(1-\mu_1 e_1)
\end{aligned}
\tag{5.17}
$$

and we have the basis 2 elements

(5.18)

where we have applied the box addition rules (5.16).

5.2. Hecke Determinants

In this section, we compute the determinant $\Theta_n^{(N)}(\beta)$ of the Gram matrix $\mathcal{H}_n^{(N)}(\beta)$ (5.7) of the basis 1 of $H_n^{(N)}(\beta)$. The result takes the form

$$
\Theta_n^{(N)}(\beta) \;=\; \prod_{m=1}^{n+1}(U_m)^{t^{(N)}_{m,n}} \tag{5.19}
$$

where $t^{(N)}_{n,m}$ are integers derived below.

In view of eqs.(5.13)(5.14), we deduce that, in terms of the matrix elements of P, the desired determinant reads

$$\Theta^{(N)}_n(\beta) \;=\; \prod_{\Lambda\in P_+}\prod_{a,b\in W^n_\Lambda} P^{-2}_{(a,b),(a,b)} \tag{5.20}$$

The diagonal terms $\pi_{(a,b)} \equiv P_{(a,b),(a,b)}$ in (5.13) satisfy the double recursion relation

$$\pi^2_{(a+\diamond_{i,m},b+\diamond_{j,\ell})} \;=\; \frac{\mu_{m+1}\mu_{\ell+1}}{\mu_m\mu_\ell}\pi^2_{(a,b)} \tag{5.21}$$

and we have the fundamental elements

$$\pi^2_{(a_0^{(n,\Lambda)},a_0^{(n,\Lambda)})} \;=\; \left(\prod_{\substack{\text{all steps}\\ v \text{ of } a_0^{(n,\Lambda)}}} w(v)\right)^2 \tag{5.22}$$

It is easy to solve (5.21)(5.22) as

$$\begin{aligned} P^2_{(a,b),(a,b)} \;&=\; \prod_{\substack{\text{all steps}\\ v,v' \text{ of } a,b}} w(v)w(v') \\ &=\; P^{-2}_{a,a}P^{-2}_{b,b} \end{aligned} \tag{5.23}$$

where, in the last line we have recognized the matrix elements of the change of basis $1 \to 2$ for the $SU(N)$ case (4.36), with w as in (4.35).

We are now ready to compute the determinant (5.20), by use of the definition (4.35). Assembling all the contributions pertaining to μ_m, we find

$$\Theta^{(N)}_n(\beta) \;=\; \prod_{m=1}^{n+1}(\mu_m)^{-\theta^{(N)}_{m,n}} \tag{5.24}$$

where

$$\theta^{(N)}_{m,n} \;=\; \sum_{p=0}^{n-1}\sum_{\Lambda'\in P_+}\sum_{1\le i<l\le N}\sum_{\substack{\Lambda\in P_+\\ \Lambda\cdot(\epsilon_l-\epsilon_i)=m}} C^{(p)}_\Lambda C^{(n-p-1)}_{\Lambda+\epsilon_i,\Lambda'}C^{(n)}_{\Lambda'} \tag{5.25}$$

This summarizes all the possible occurrences of μ_m in (5.20). We have denoted by $C^{(r)}_{\Lambda,\Lambda'}$ the number of paths of r steps on Π_+, from Λ to Λ', which reads

$$C^{(r)}_{\Lambda,\Lambda'} \;=\; \sum_{\sigma\in S_N}(-1)^{l(\sigma)}D^{(r)}_{\Lambda'-\sigma(\Lambda)} \tag{5.26}$$

where the necessary reflections (additions/subtractions) have been performed on the paths on Π from Λ to Λ'. The combination of C's in (5.25) stands for the total number of pairs $a, b \in W^n_{\Lambda'}$, with one specified edge $(\Lambda, \Lambda+\epsilon_i)$. The edge indeed separates one of the walks a or b into two parts: (i) the portion between ρ and Λ, of length p (a total of $C^{(p)}_{\Lambda}$ paths) (ii) the portion between $\Lambda+\epsilon_i$ and Λ', of length $n-p-1$ (a total of $C^{(n-p-1)}_{\Lambda+\epsilon_i,\Lambda'}$ paths). The extra factor accounts for the $|W^n_{\Lambda'}| = C^{(n)}_{\Lambda'}$ possibilities for the other walk.

The desired formula (5.19) then follows from the definition (2.26), with $t^{(N)}_{m,n} = \theta^{(N)}_{m,n} - \theta^{(N)}_{m+1,n}$. The first few numbers $t^{(N)}_{m,n}$ are listed in the case of $N=3$ in Table VIII.

$m\backslash n$	1	2	3	4	5	6	7	8	9	10
1	0	1	5	21	85	331	1155	2688	-7872	-196425
2	1	2	6	26	136	774	4599	28080	174951	1108158
3		1	5	22	102	521	2933	17872	115344	774396
4			1	10	69	424	2528	15184	93537	595602
5				1	17	171	1395	10305	72513	499291
6					1	26	358	3746	33889	281728
7						1	37	666	8666	94096
8							1	50	1137	17952
9								1	65	1819
10									1	82
11										1

Table VIII: The powers $t^{(3)}_{m,n}$ of U_m in the Hecke meander determinant $\Theta^{(3)}_n(\beta)$, for $n = 1, 2, ..., 10$. The determinant of the matrix $\mathcal{H}^{(3)}_3(\beta)$ of (5.8) is read in the third column to be $\Theta^{(3)}_3(\beta) = U_1^5 U_2^6 U_3^5 U_4$.

6. Conclusion

6.1. Generalized meanders

In this paper, we have investigated two possible directions of generalization of the notion of meander. The first direction, developed in Sects.3 and 4, defines the $SU(N)$ meanders of order Nn as pairs (a,b) of $SU(N)$ walk diagrams of Nn steps; to these

objects we have associated the quantity $\big((a)_1,(b)_1\big)$, namely the scalar product of the two corresponding basis 1 elements of the ideal $\mathcal{I}_{Nn}^{(N)}(\beta)$. This quantity however has a simple combinatorial interpretation only in the $SU(2)$ case, where it relates directly to the number of connected components of the meander (see eq.(2.11)). Unfortunately, we have not yet been able to find a good combinatorial interpretation for $N\geq 3$, such as formulations as (polymer or membrane) folding problems for instance. We intend to return to this aspect in a later publication.

The second possible direction, developed in Sect.5, would rather define meanders as pairs of *couples* of open $SU(N)$ walk diagrams of n steps ending at some weight $\Lambda\in P_+$, the Weyl chamber of $sl(N)$. Remarkably, the two pictures coincide for $N=2$, thanks to the existence of an isomorphism between the left ideal $\mathcal{I}_{2n}^{(2)}(\beta)=H_{2n}^{(2)}(\beta)e_1e_3...e_{2n-1}$ and the Temperley-Lieb algebra $H_n^{(2)}(\beta)$. Schematically, this is due to the two equivalent formulations of a walk diagram of order $2n$ as a path $a\in W_{2n}^2$ from the origin 1 to itself, or the pair formed by its first and second halves (respectively a path of n steps say from the origin 1 to the weight m, and from the weight m to the origin 1), which, up to reversal of the second half, form a pair $(a',b')\in W_m^n$. This breaks down for $N\geq 3$, as in the pair $(a',b')\in W_\Lambda^n$ the "return path" b' has to be described in the reverse order, from Λ^t to $\rho^t=\rho$, and we cannot identify the pair with a walk diagram of $2n$ steps, starting and ending at the origin ρ (there will be in general a necessary jump from Λ to Λ^t, or alternatively a necessary reversal of all directions on Π_+ for the return path).

The results of Sects.3,4 however seem to suggest that the first generalization is the good one, as the results for the meander determinants take simple generalized forms, which we could not find for the second generalization of Sect.5.

6.2. Generalized semi-meanders

The study of the whole Hecke quotient $H_n^{(N)}(\beta)$ has the advantage of offering a better framework to generalize the notion of semi-meander, introduced in [7] [8] [18]. The original (multi-component) semi-meander problem is that of enumerating the topologically inequivalent configurations of a (several) nonselfintersecting loop(s), crossing a half-line through n given points. Any such configuration is called a (multi-component) semi-meander of order n. In comparison with the meander case, the novelty is that loops can freely wind around the origin of the half-line. The winding number is then defined as the minimal number of intersections which would be created by replacing the half-line with a line, plus one (this one was not added in the definition of the winding used in [7], it permits however to

present a more unified notion when discussing generalizations). Considering that the line separates the semi-meander into an upper and lower “open” arch configurations it is easy to interpret any semi-meander with winding m as a pair of open walk diagrams of order n ending at the weight m, or equivalently with a basis 1 element of the Temperley-Lieb algebra $H_n^{(2)}(\beta)$, corresponding to a walk diagram of order $2n$ with middle weight $\lambda_n = m$.

This suggests the following generalization of semi-meanders. We will call $SU(N)$ semi-meander of order n with winding $\Lambda \in P_+$ any pair of open walk diagrams $a, b \in W_\Lambda^n$. The semi-meander matrix for order n and winding Λ is then defined, using the basis 1 of $H_n^{(N)}(\beta)$ described in Sect.5, as the Gram matrix with entries

$$\big[\mathcal{H}_\Lambda^{(n)}(\beta)\big]_{a,b} \;=\; \mathrm{Tr}\big((a,b)_1\big) \quad a, b \in W_\Lambda^n \tag{6.1}$$

Note that when $\Lambda = \rho$ (only possible if the order n is of the form Nk for some integer k), this matrix is identical to the $SU(N)$ meander matrix (4.17), which suggests to interpret a semi-meander with winding ρ as a meander. This is of course a consequence of the identification $W_\rho^{Nk} \simeq W_{Nk}^{(N)}$ between the open walk diagrams ending at the origin and the walk diagrams of same order.

In [11], we have derived a simple formula for the determinant of these matrices, when $N = 2$. The strategy used was again a direct orthogonalization of the basis 1 of the corresponding vector space, leading to a basis 2’ strictly distinct from the restriction of the basis 2 of $H_n^{(2)}(\beta)$. In the general $N \geq 3$ case, we expect the determinant of (6.1) to still be given by some simple product formula involving the Chebishev polynomials (2.14).

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References

[1] J. Touchard, *Contributions à l'étude du problème des timbres poste*, Canad. J. Math. **2** (1950) 385-398.

[2] W. Lunnon, *A map–folding problem*, Math. of Computation **22** (1968) 193-199.

[3] K. Hoffman, K. Mehlhorn, P. Rosenstiehl and R. Tarjan, *Sorting Jordan sequences in linear time using level-linked search trees*, Information and Control **68** (1986) 170-184.

[4] V. Arnold, *The branched covering of $CP_2 \to S_4$, hyperbolicity and projective topology*, Siberian Math. Jour. **29** (1988) 717-726.

[5] S. Lando and A. Zvonkin, *Plane and Projective Meanders*, Theor. Comp. Science **117** (1993) 227-241, and *Meanders*, Selecta Math. Sov. **11** (1992) 117-144.

[6] K.H. Ko, L. Smolinsky, *A combinatorial matrix in* 3*-manifold theory*, Pacific. J. Math **149** (1991) 319-336.

[7] P. Di Francesco, O. Golinelli and E. Guitter, *Meander, folding and arch statistics*, to appear in Math. and Comp. Modelling (1997), hep-th/950630.

[8] P. Di Francesco, O. Golinelli and E. Guitter, *Meanders and the Temperley-Lieb algebra*, to appear in Comm. Math. Phys. (1997), hep-th/9602025.

[9] H. Temperley and E. Lieb, *Relations between the percolation and coloring problem and other graph-theoretical problems associated with regular planar lattices: some exact results for the percolation problem*, Proc. Roy. Soc. **A322** (1971) 251-280.

[10] P. Martin, *Potts models and related problems in statistical mechanics*, World Scientific (1991).

[11] P. Di Francesco, *Meander determinants*, preprint UNC-CH-MATH-96/1 (1996) hep-th/9612026.

[12] N. Reshetikhin, *Quantized universal enveloping algebras, the Yang-Baxter equation and invariants of links 1 and 2*, LOMI preprints E-4-87 and E-17-87 (1988).

[13] M. Jimbo, T. Miwa and M. Okado, *An $A_{n-1}^{(1)}$ family of solvable lattice models*, Mod. Phys. Lett. **B1** (1987) 73-79, and *Solvable lattice models related to the vector representation of classical simple Lie algebras*, Comm. Math. Phys. **116** (1988) 507-525.

[14] R. Dipper and G. James, *Representations of Hecke algebras of general linear groups*, Proc. London Math. Soc. **52** (1986) 20-52; *Blocks and idempotents of Hecke algebras of general linear groups*, Proc. London Math. Soc. **54** (1987) 57-82.

[15] I. Cherednik,*A new interpretation of Gelfand-Tzeitlin bases*, Duke Math. Jour. **54** (1987) 563-577.

[16] G. Murphy, *On the representation theory of the symmetric groups and associated Hecke algebras*, Jour. Algebra **152** (1992) 492-513.

[17] A. Kuniba and T. Nakanishi, *Level-rank duality in fusion RSOS models*, in Proc. of the international colloquium on modern quantum field theory, Bombay, Eds S. Das et al., World Scientific (1991).

[18] P. Di Francesco, O. Golinelli and E.Guitter, *Meanders: a direct enumeration approach*, Nucl. Phys. **B482[FS]** (1997) 497-535.